## THE FORTY-NINTH REGULAR MEETING OF THE SAN FRANCISCO SECTION

The forty-ninth regular meeting of the San Francisco Section of the Society was held at the University of Washington on Saturday, June 12, 1926. In the absence of the Chairman, Professor Blichfeldt, Professor Daniel Buchanan was elected temporary Chairman. The total attendance was twenty, including the following fifteen members of the Society:

E. T. Bell, Bernstein, Daniel Buchanan, A. F. Carpenter, Cramlet, Dines, Gavett, Hanawalt, Jerbert, McAlister, W. E. Milne, Mullemeister, Neikirk, Rojansky, Winger.

Following the meeting, the members of the Section at the University of Washington acted as hosts to the visiting members and their friends at a luncheon and an excursion on Puget Sound.

Abstracts of the papers presented at this meeting follow below. The papers by Brown, Buchanan (first paper), Cleveland, Gage, Gehman, Johnston, D. H. Lehmer, D. N. Lehmer, and Whyburn were read by title. Mr. Brown, Mr. Gage, and Miss Johnston were introduced by Professor Buchanan, and Mr. Cleveland by Professor R. L. Moore.

1. Professor E. T. Bell: *The modular Bernoullian and Eulerian functions.*

This paper, to appear shortly in the publications in science of the University of Washington, is a brief prospectus concerning a remarkable new class of polynomials in two complex variables. In one of the complex variables they are Appell polynomials; in the other they satisfy no similar algebraic differential equation. The theory includes as very special cases those of the Bernoulli and allied numbers, the anharmonic polynomial generalizations of these devised by the author, and the common Bernoullian and Eulerian functions. The general theory is more readily developed than is that of any of its special cases, because in the latter the underlying symmetry, which is that of the linear group abstractly identical

with the cross-ratio group, is obscured owing to coincidences of two or more values of the complex variables at the same point.

2. Professor E. T. Bell: *Irregular fields.*

An irregular field $IF$ differs from an abstract field $F$ only in that division by $m>1$ "irregular elements" in $IF$ is not permissible (instead of the unique zero element in $F$). The postulates for $IF$ are stated in abstract form and then *solved*, giving several interconnected $IF$'s. Finally, the abstract theory is applied to the algebra of the elliptic polynomials—a new class of polynomials including as degenerate cases the Bernoullian, Eulerian, and their anharmonic generalizations. Differential and difference equations in $IF$ are discussed and applied.

3. Professor E. T. Bell: *On a fundamental theorem in the theory of class-number relations.*

This paper will appear in full in an early issue of this BULLETIN.

4. Professor B. A. Bernstein: *On the serial relation in Boolean algebras.*

This paper appears in full in the present issue of this BULLETIN.

5. Miss C. I. Johnston: *Isosceles-triangle solutions when the two finite bodies are oblate spheroids.*

When two finite spheres move in circles about their common center of gravity, an infinitesimal third body will describe a periodic orbit, if given the proper initial projection. This orbit is the straight line through the center of gravity of the finite bodies and perpendicular to the plane of their motion. This solution is called the isosceles-triangle solution. The present paper determines the corresponding isosceles-triangle solution when the finite bodies are similar oblate spheroids of equal mass. It will appear in the TRANSACTIONS OF THE ROYAL SOCIETY OF CANADA.

6. Mr. J. F. Brown: *Asymptotic orbits near the straight-line equilibrium points in the problem of three bodies when two of the bodies are oblate spheroids.*

Warren and Strömgren determined, independently, the orbits which are asymptotic to the straight-line equilibrium points, when two of the bodies are finite spheres moving in circles, and the third body is infinitesimal. This paper deals with the corresponding orbits when the finite bodies are oblate spheroids. It will appear in the next issue of the TRANSACTIONS OF THE ROYAL SOCIETY OF CANADA.

7. Professor Daniel Buchanan: *Isosceles-triangle orbits of the second genus.*

Poincaré has shown the existence of two types of periodic orbits in the problem of three bodies. These he called orbits of the first and of the second

genus. If the first-genus orbits have the period $T$, then the second-genus orbits will have the period $k(T+p)$, where $k$ is an integer and $p$ is a parameter. This paper deals with the second-genus orbits near one of the isosceles-triangle solutions previously determined by the author; it will appear in the TRANSACTIONS OF THE ROYAL SOCIETY OF CANADA.

8. Professor Daniel Buchanan: *Periodic orbits of the second genus near the straight-line equilibrium points.*

Moulton's "oscillating satellite" exhibits periodic orbits near the straight-line equilibrium points in the problem of three bodies. These orbits are of the first genus. This paper determines the second-genus orbits in the vicinity of Moulton's solutions. Along with the preceding paper, it is the first to exhibit orbits of the second genus.

9. Mr. W. H. Gage: *An illustrative orbit of the second genus.*

The earlier steps of the integrations in the preceding paper are carried out algebraically. The approximate second-genus orbits are determined for certain assigned values of the masses, initial displacements, etc.

10. Professor A. F. Carpenter: *Intersector sequences.*

On each of an ordered set of three ruled surfaces $R_1$, $R_2$, $R_3$, whose rulings $r_1$, $r_2$, $r_3$ correspond in sets of three, one from each surface, there lies a one-parameter family of curves whose tangents, called intersector tangents of the surface, are such that those tangent to $R_1$, along $r_1$, cut $R_2$ along $r_2$, those tangent to $R_2$ along $r_2$ cut $R_3$ along $r_3$, etc. The paper defines certain properties of sequences of these intersector tangents and establishes a theorem involving a general invariant.

11. Professor A. F. Carpenter: *Closed systems of invariants in projective differential geometry.* Preliminary communication.

A system of ten invariants in the projective differential theory of triads of ruled surfaces is shown to be closed with reference to two different operations.

12. Mr. C. M. Cramlet: *Applications of the determinant and permanent tensors to determinants of general class and allied tensor functions.*

With a given set of fundamental tensors, tensor functions can be expressed by the tensor $\gamma^{r_1 \cdots r_n}_{s_1 \cdots s_n}$ (see this BULLETIN, vol. 32, p. 213). Determinants and permanents are examples of such functions. Theorems for permanents formally analogous to the theorems for transformations and multiplication of determinants of a rank 2 tensor are given. By the use of

the gamma tensor and its specializations the determinant and permanent tensors, insight into the properties of determinants of higher class is obtained. In particular, the rule for file multiplication appears to be applicable to more general types of functions than have been considered heretofore.

13. Professor L. L. Dines: *Note on certain associated systems of linear equalities and inequalities.*

In this note the author presents some theorems complementary to those of his paper in the ANNALS OF MATHEMATICS (1925).

14. Professor L. L. Dines: *General systems of vectors.*

A set of $m$ real numbers $(a_1, a_2, \cdots, a_m)$ may be interpreted as the components of a vector in space of $m$ dimensions. A set of $m$ real functions $(f_1(p), f_2(p), \cdots, f_m(p))$ of a variable $p$ may therefore be interpreted as a system of vectors, each value of the parameter $p$ determining a vector of the system. In the present paper the author considers certain properties of a *general* system of vectors, the generality consisting in the fact that the range of the parameter $p$ is entirely unconditioned. The methods and results generalize some of those contained in earlier papers of the author.

15. Professor L. L. Dines: *On linear integral inequalities.*

This investigation is a result of an effort to generalize to functions on a general range, the theory developed by the author for systems of linear algebraic inequalities. Peculiar difficulties are inherent in the case of continuous functions of a continuous variable, and it therefore seemed best to consider this case alone. The study of the integral inequality $\phi(x) - \int_a^1 \alpha(x,t)\phi(t)dt > 0$ involves the study of the functions orthogonal to a given set of functions $\psi_1(x), \psi_2(x), \cdots, \psi_n(x)$, in particular the conditions under which such a function may be everywhere positive.

16. Mr. C. M. Cleveland: *Concerning cut points of a continuous curve.*

The author establishes the following theoems: I. *Every connected set of cut points of a continuous curve is arcwise connected.* II. *Every maximal connected set of cut points of a continuous curve is an acyclic continuous curve.*

17. Dr. H. M. Gehman: *Some relations between a continuous curve and its subsets.*

In this paper, the author proves that if $M$ is a continuous curve, a necessary and sufficient condition that every sub-continuum of $M$ be a continuous curve, is that every irreducibly connected set between two points of $M$ is an arc. Either of the following conditions is sufficient in order that every subcontinuum of a *continuum* $M$ be a continuous curve: (1) given any positive number $e$, $M$ contains at most a finite number of

mutually exclusive continua of diameter greater than $e$; (2) given any positive number $e$, $M$ can be expressed as the sum of a finite number of continua each of diameter less than $e$, and each pair having at most a finite number of points in common. It is shown that certain other conditions which are sufficient when $M$ is assumed to be a continuous curve, are not sufficient under the weaker assumption that $M$ is a continuum.

18. Mr. D. H. Lehmer: *Note on solving the indeterminate equation* $t^2-p^2Du^2=1$.

The solution of the Pellian equation in which the argument $p^2D$ contains a square factor can be conveniently obtained from the solution of the equation for the argument $D$. The method depends on discovering among the multiple solutions of the latter equation, one in which the $u$ is divisible by $p$. Several theorems are given which limit the number of trials in searching for this solution. The simple case in which $p=2$ seems to have been noticed by Legendre. This method would be useful in extending the tables of the solutions of the Pellian equation, since it obviates the necessity of expanding the square root of the argument in a continued fraction.

19. Mr. D. H. Lehmer: *Note on the Mersenne number* $2^{139}-1$.

This paper appears in the present issue of this BULLETIN.

20. Professor D. N. Lehmer: *Further results in the congruential theory of magic squares.* Second preliminary paper.

The author has reduced the theory of uniform step magic squares (odd order) to the theory of two simultaneous congruences involving the greatest integer function. These congruences involve essentially four arbitrary constants, and the various kinds of magic squares are obtained by subjecting these constants to certain conditions. His latest results are connected with nuclear squares.

21. Professor L. I. Neikirk: *Some finite linear non-associative algebras.* Second paper.

The present paper is a generalization and extension of a previous paper on the same subject (see this BULLETIN, vol. 19, p. 195). A complete set of postulates is given and a development of the general theory. Three kinds of powers are possible: $(S(\cdots(S(S(SS)))\cdots))$, $(\cdots(((SS)S,\cdots)S)$, $(\cdots((S(SS))S)\cdots)$. Two examples of these algebras are exhibited along with multiplication tables. Both papers will be published in THE UNIVERSITY OF WASHINGTON STUDIES, mathematical series, number 3.

22. Mr. G. T. Whyburn: *Concerning the cut points and end points of continua.*

If $K$ and $H$ denote the set of all the cut points and end points respectively of a continuum $M$, then (1) every bounded subcontinuum of

$K+H$ is an acyclic continuous curve; (2) if $T$ is a subcontinuum of $K$, and $Z$ is any connected subset of $M-T$, then $T$ contains at most one limit point of $Z$; (3) every subcontinuum of $K$ contains a subcontinuum belonging to the boundary of some single complementary domain of $M$; (4) $M$ is connected im kleinen at every point of $H$ which is accessible from $S-M$; (5) if $M$ is irreducible between some pair of points, then it is connected im kleinen at every point of $H$; (6) no point of $H$ is a boundary point of more than one complementary domain of $M$; (7) in order that a point $P$ of $M$ should belong to $K$, it is sufficient that it should be a cut point of the boundary of some complementary domain of $M$; (8) a point of a continuous curve $M$ is an end-point if it is not interior to any arc of $M$.

23. Mr. G. T. Whyburn: *Concerning continuous curves.*

R. L. Moore has shown that a *continuous curve* $M$ is an acyclic continuous curve if and only if $M$ is not disconnected in the weak sense by any subset of all the non-cut points of $M$. The author shows that this proposition holds for any *bounded continuum*. It is also shown that if $K$, $H$, and $N$ respectively denote the set of all the cut points, end points, and simple closed curves of a continuous curve $M$, then (a) $H$ is totally disconnected; (b) $K+H+N=M$; (c) $KN$ is countable; (d) if $T$ is any connected subset of $K$, then $\bar{T}$ is an acyclic continuous curve if and only if $T$ belongs to $K+H$; (e) $K$ does not contain more than a finite number of mutually exclusive continua of diameter greater than a given positive number; (f) if $L$ is any connected subset of $M-N$, then $L$ is arcwise connected and $\bar{L}$ is an acyclic continuous curve; (g) if the point $P$ belongs to $M$ but not to the boundary of any complementary domain of $M$, then $M$ contains a simple closed curve enclosing $P$ and *of diameter arbitrarily small.* (The italicized portion of (g) was established by C. M. Cleveland.)

24. Professor E. H. McAlister: *A new method of determining azimuth.*

In contrast with existing methods, the peculiar merit of the method described is that it does not require any knowledge either of the latitude or local time. The observations and computations are both simple and brief. The accuracy attainable is nearly the same as the accuracy with which the observer can measure the horizontal angle between Polaris and the azimuth mark, all other small errors of observation being reduced in the ratio of one to thirty or better. Certain quantities, derived from the apparent positions of the stars, have been tabulated at ten-day intervals for the quarter century 1925–1950. The method is available at any season, at any hour of darkness, and in any latitude in which Polaris can be conveniently observed. The method has been generalized, with some loss of simplicity in computation, but with greater freedom in the choice of stars.

B. A. BERNSTEIN,
*Secretary of the Section.*

# THE BOREL THEOREM AND ITS GENERALIZATIONS*

BY T. H. HILDEBRANDT

That any one should attempt to devote a paper to the subject of the Borel Theorem may at first glance seem a presumption. A brief investigation will however reveal the following facts. (a) The Borel Theorem is closely related to the fundamental postulates of linear order. (b) There are many extensions and analogs of the Borel Theorem, some of which are hidden away in papers on other subjects. (c) The Borel Theorem has held and still holds a central position in the development and analysis of general spaces.

The arrangement of topics in the paper is suggested by the previous paragraph. No claim is made for completeness with respect to the extensions and analogs of the Borel Theorem, due to the nature of the case. Nor do I claim any originality in the material presented. I hope that a systematic treatment of the Borel Theorem in general spaces will be suggestive and perhaps create further desirable interest and results in these spaces.

## I. The Borel Theorem and its Extensions for the Linear Interval and $n$-Dimensional Space

In order to give a simple statement of the Borel Theorem I shall use the phrase "a family $\mathfrak{F}$ of intervals $I$ covers the point-set $E$" to mean that every $x$ of $E$ is interior to some interval $I_x$ of $\mathfrak{F}$. Then the Borel Theorem in its simplest form may be stated as follows.

*If the family $\mathfrak{F}$ of intervals $I$ covers the closed interval $(a, b)$ then a finite subfamily of $\mathfrak{F}$ covers $(a, b)$.*

* An address presented at the request of the program committee before the joint meeting of this Society and the American Association for the Advancement of Science, Section A, at Kansas City, December 30, 1925.

1. *Historical Note.** As in the case of other important mathematical results, the conception of the Borel Theorem is an interesting chapter in mathematical history, to which the future will no doubt add contributions. So far the honor of having first stated an outright theorem on intervals belongs to Borel,† though Borel's formulation deals only with the reduction of a denumerable family of intervals to a finite one. The name Heine-Borel seems to be due to Schoenfliess,‡ who noted the relationship of the Borel Theorem to Heine's proof of the uniform continuity of a function continuous on a closed interval, published in 1872.§ That Heine was aware of the fact that an interval theorem lay hidden away in his proof seems rather doubtful. As a matter of fact priority on the uniform continuity seems to go back at least to G. Lejeune-Dirichlet, though he suffered the penalty of not publishing his result immediately. A proof of the theorem almost identical with that of Heine appears in an exposition of his lectures given in 1854 brought out in 1904 by G. Arendt.|| Another result carrying within it the germs of the Borel Theorem is due to S. Pincherle,¶ who gave the following theorem in 1881.

*If the positive-valued function $f(x)$ is bounded from zero in some neighborhood of each point of a closed interval, then there exists an $e$ such that*

$$f(x) > e > 0$$

*for all points of the interval.*

He remarks that this theorem can be made the basis for proof of uniform continuity and of the uniform convergence

* Cf. ENZYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN, vol. $II_1$, pp. 882 et seq.

† Paris thesis, 1894, p. 43; ANNALES DE L'ÉCOLE NORMALE, (3), vol. 12 (1895), p. 51.

‡ *Bericht über die Mengenlehre*, JAHRESBERICHT DER VEREINIGUNG, vol. 8 (1900), pp. 51 and 119. In a later edition (Schoenfliess-Hahn, *Entwickelungen*, vol. I (1913), p. 235), he reverts to the name Borel Theorem.

§ Cf. JOURNAL FÜR MATHEMATIK, vol. 74 (1872), p. 188.

|| *Vorlesungen über die Lehre von den bestimmten Integralen*, Braunschweig, 1904.

¶ MEMORIE DI BOLOGNA, (4), vol. 3 (1881), pp. 151ff.

of a series of functions, uniformly convergent at every point of a closed interval.

To whom shall go the honor of first having conceived of the possibility of extending the Borel theorem to the case where the given family of intervals is not necessarily denumerable is another debatable question. In a way it is true that Dirichlet, Heine, and Pincherle were dealing with this case. Closely related is also the following theorem for the plane, due to P. Cousin.*

*If to each point of a closed region there corresponds a circle of finite radius, then the region can be divded into a finite number of subregions such that each subregion is interior to a circle of the given set having its center in the subregion.*

Lebesgue, to whom this extension is usually credited, claims to have known the result in 1898,† but first published it in 1904 in his *Leçons sur l'Intregation*. W. H. Young‡ published a proof in 1902. As a matter of fact the statement and proof of the Borel Theorem given by Schoenfliess in his 1900 Bericht can easily be interpreted to be that of the extension in question.

In considering these divergent claims it seems simplest and most just to call the theorem the Borel Theorem, and in case a distinction is necessary indicate it as the denumerable-to-finite or any-to-finite form. Moreover the theorem which gives the reduction from any set of intervals to an equivalent denumerable family due to Lindelöf in a general space proves to be only another case of a more general Borel Theorem.

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* ACTA MATHEMATICA, vol. 19 (1895), p. 22; Fréchet: CONGRÈS SOCIÉTÉS SAVANTES, 1924, p. 68.

† For his attitude on the priority question, see BULLETIN DES SCIENCES MATHÉMATIQUES, (2), vol. 31 (1907), pp. 132–4.

‡ PROCEEDINGS OF THE LONDON SOCIETY, vol. 35 (1902), pp. 384–8.

§ Cf. Schoenfliess-Hahn, *Entwickelungen der Mengenlehre*, vol. 1, 1913, pp. 235 et seq.

‖ Cf. Borel (Baire), COMPTES RENDUS, vol. 140 (1905), p. 299; Capelli, NAPOLI RENDICONTI, (3), vol. 15 (1909), p. 151.

2. *Proofs of the Borel Theorem.*§ (a) *By Successive Subdivisions.*|| Of the proofs of the Borel Theorem, perhaps the simplest in form is the one which proceeds by successive subdivisions. It is an indirect proof. Assume that the theorem is not true for the interval $(a, b)$. Then if we divide the interval into two or more equal parts, it will be not true for one of these parts. This process applies indefinitely, and we have a sequence of closed intervals, each containing the succeeding, with lengths converging to zero, giving a single point $x$ of $(a, b)$ common to the intervals. This point being interior to an interval $I$ of the family $\mathfrak{F}$, $I$ will contain the intervals of the sequence after a certain stage, thus yielding a contradiction.

We observe that in a way this proof connects the Borel Theorem with the Cantor Theorem: *An infinite sequence of closed sets of points, each containing the succeeding, have a common point.*

(b) *By Use of the Weierstrass-Bolzano Theorem.* Another method of applying a similar process is to note that if the theorem is not true for $(a, b)$ then if we divide the interval into $n$ equal parts, it will be not true for one of these parts. Let $(a_n, b_n)$ be the interval of length $(1/n)(b-a)$ for which the theorem does not hold, and $x_n$ a point belonging to this interval. Then by the Weierstrass-Bolzano theorem, since the sequence $\{x_n\}$ is bounded, there will exist a subsequence $\{x_{n_m}\}$ having as limit the point $x_0$ of the interval $(a, b)$. If $x_0$ is interior to $I$ of the family $\mathfrak{F}$, then, by the properties of limits of sequences, we find that one of the intervals $(a_n, b_n)$ is interior to $I$.

This method of proof is not quite so simple nor so elegant as the first. It uses the Weierstrass-Bolzano theorem, which itself is usually derived by a process of successive subdivisions.

(c) *Direct Proof by Subdivisions.* The two preceding proofs on account of their indirect character do not give a method for actually selecting the intervals required by

the theorem. We can however use the subdivision idea to make such a selection. Let $d_x$ be the least upper bound of the values of $d$ for which the interval $(x-d, x+d)$ is interior to some interval of $\mathfrak{F}$. Then the function $d_x$ is bounded from zero on the closed interval $(a, b)$, i. e., there exists a $d$ such that for all $x$ of $(a, b)$ $d_x > d > 0$. This can be deduced from the fact that $d_x$ is a continuous function on a closed interval, or connected with the two results (a) that there exists for each $x$ a vicinity of $x$ such that $d_x$ is bounded from zero in this vicinity, and (b) that for any function defined on $(a, b)$ there exists a point $x$ such that the greatest lower bound of $f$ for each vicinity of $x$ is the same as the greatest lower bound of $f$ on $(a, b)$. The final step in the proof then is to divide $(a, b)$ into intervals of length less than $d$, each of which will be covered by some member of the family $\mathfrak{F}$ by the definition of $d_x$.

Observe that in this case we replace all the intervals to which a point is interior by a single member of another family, a family which is really involved in the proof of the uniform continuity theorem as given by Heine and by Dirichlet. The Pincherle result is also involved in the preceding proof, as well as the linear analog of Cousin's formulation of the Borel Theorem in the plane.* We might call attention to the following almost self-evident result involved in the preceding proof.

*If every point of a bounded set $E$ is the middle point of an interval of length $2d_x$, and $d_x$ is bounded from zero, then all the points of $E$ are interior to a finite number of these intervals.*

(d) *Denumerable-to-Finite.*† When the given family is denumerable it is possible to select the finite family as follows. Suppose the intervals of $\mathfrak{F}$ are arranged in se-

* Cf. also Baire, ANNALI DI MATEMATICA, (3), vol. 3 (1899), pp. 13–15; Borel, COMPTES RENDUS, vol. 140 (1905), p. 299; Wirtinger, WIENER BERICHTE, vol. 108, IIa, pp. 1242–3.

† Cf. Young, W. H., PALERMO RENDICONTI, vol. 21 (1906), p. 127; Fréchet, PALERMO RENDICONTI, vol. 22 (1906), pp. 22–23.

quential order $I_1, I_2, \cdots, I_n, \cdots$. By a step-by-step process, we obtain an equivalent series, retaining those intervals which contain at least one point not interior to the preceding intervals of the array. Let $I_1, I_{n_2}, \cdots, I_{n_m}, \cdots$ be the members of the retained family. If this is finite, the theorem is proved. In the contrary case, we have an element $x_m$ interior to $I_{n_m}$ but not to the preceding intervals. By the use of the Weierstrass-Bolzano Theorem, this sequence contains a subsequence $x_k'$ having a limit $x_0$ of $(a, b)$. The contradiction arises from the fact that the interval $I_{n_m}$ containing $x_0$ as an interior point will contain an infinite number of the $x_k'$ as interior points, i. e., members of the sequence $\{x_m\}$ of index higher than $n_m$. We shall see later in the present paper that this method of proof is effective in general spaces.

(e) *Borel's Proof. The Lebesgue Chain.* The first proof of Borel also contains a scheme for selecting the finite subfamily, but it rests upon the properties of Cantor ordinal numbers, and the denumerability of the family, or of a family connected with the given family. Starting with the point $a$, there exists an interval $I_1$ containing $a$ as interior point. If $b_1$ is the left hand end-point of $I_1$, let $I_2$ contain $b_1$ as interior point. Continuing thus, we get a sequence of intervals $I_1, I_2, \cdots, I_n, \cdots$. If the point $b$ has not been reached, we get a limiting point $b_\omega$ of the left hand end-points $b_n$ and an interval $I_\omega$ containing $b_\omega$. By repeating the process, since the set of intervals is denumerable, we must eventually include the point $b$ with an interval $I_\alpha$ where $\alpha$ is a transfinite ordinal of the second kind. We now extract from this well-ordered set of intervals a finite subset by noting that any interval $I_\alpha$ associated with a limit number $\alpha$ includes an infinite number of the end-points $b_\beta$ of intervals preceding it. This enables us to select a decreasing array of ordinal numbers and such a decreasing set is finite.

This proof hinges on two facts, (a) that the system of intervals is denumerable, and (b) that a decreasing set of

ordinal numbers selected from a well-ordered increasing set is finite.

While the denumerability of the family $\mathfrak{F}$ is postulated in this proof, the denumerability of the array $I_1, \cdots, I_\omega, \cdots$ leading to the point $b$ can be deduced from the following fundamental theorem on intervals.

*Any family of non-overlapping intervals is denumerable.*

For the interval (0, 1), this follows from the fact that there exist at most $n$ intervals of the set having a length greater than $1/n$, which gives a system for enumeration. Since the unbounded straight line can be broken up into a denumerable set of intervals of length unity, the extension of this result to any family of non-overlapping intervals is immediate.

In the preceding proof of the Borel Theorem the intervals $(b_\alpha, b_{\alpha+1})$ form a nonoverlapping family, which is therefore denumerable. Then the assumption that $b$ would not be reached by a denumerable set of steps, would lead to a contradiction. In point of fact, there is contained implicitly in these considerations a theorem which has been called the Lebesgue Chain Theorem* which has been used by him in the discussion of lengths of curves. The theorem may be stated as follows.

LEBESGUE CHAIN THEOREM. *If the family $\mathfrak{F}$ of intervals $I$ is such that to every point $x$ of an interval $(a, b)$ excepting perhaps $b$, there correspond intervals of the family having $x$ as left-hand end-point, then there exists a finite or denumerable subfamily of these intervals without common points, containing each point of the interval, excepting possibly $b$, as an interior point or left-hand end-point.*

We shall give another proof of this theorem later on. Also we shall see that the idea of a Lebesgue chain and of applying a process of reduction via a decreasing set of ordinals underlies the proofs of other theorems similar to the Borel Theorem.

* *Leçons sur l'Intégration*, p. 63.

(f) *Dedekind-Cut Proof.** Finally we call attention to a proof which depends on the Dedekind-Cut Axiom, i. e., is based on the properties of linear order. We observe that the point $a$ is interior to an interval $I$ of the family $\mathfrak{F}$. Let $x'$ be the least upper bound of the points of $(a, b)$ which can be reached with a finite number of intervals starting from $a$, i. e., $x'$ is defined by a Dedekind cut. Now $x'$ will belong to the interval $(a, b)$ which is closed. Consequently there will be an interval $I_{x'}$ of the family $\mathfrak{F}$ to which $x'$ is interior. It follows that $x'$ is the point $b$.

An analysis of the proof shows that it rests upon the following principle.

INDUCTION PRINCIPLE FOR LINEAR ORDER. *Suppose a statement $S$ satisfies the following conditions relative to an interval $(a, b)$ (which may be the infinite interval $(-\infty, +\infty)$):* (1) *there exists a point of the interval for which $S$ is true,* (2) *if the statement $S$ is true for all points preceding $x'$, then there exists a point $y$ beyond $x'$ for which $S$ is true. Under these conditions $S$ holds for the entire interval $(a, b)$.*

A. Khintchine† has pointed out that this Induction Principle is logically equivalent to the Dedekind-Cut Axiom.

That the Dedekind-Cut Axiom implies the Induction Principle is practically contained in the above proof of the Borel Theorem. On the other hand, assume the truth of the Induction Principle. Suppose we have divided the points of the closed interval $(a, b)$ into two groups $A$ and $B$, each containing at least one point, and such that every point of $A$ is less than (precedes) every point of $B$, and that $A$ and $B$ contain all the points of $(a, b)$. Let the statement $S$ be "The point $x$ is a member of $A$." Since the conclusion of the Induction Principle is not holding, it

* Lebesgue, *Leçons*, p. 105; O. Veblen, this BULLETIN, vol. 10 (1904), pp. 436–9 (see also TRANSACTIONS OF THIS SOCIETY, vol. 6 (1905), p. 167, where it is pointed out that the Borel Theorem applies to any well-ordered set); F. Riesz, COMPTES RENDUS, vol. 140 (1905), pp. 244–6.

† FUNDAMENTA MATHEMATICAE, vol. 4 (1922), pp. 164–6.

follows that either condition (1) or (2) is not holding. Now (1) is true, hence (2) must be false, i. e., there exists a point $x'$ such that for every $x$ preceding it the statement $S$ is true, but $S$ is not true for any point following $x'$. Since $A$ and $B$ contain all points of $(a, b)$ it follows that the point $x'$ belongs to one of these classes, i. e., is either the maximum of the class $A$, or the minimum of the class $B$.*

Along the same line is the observation that the Borel Theorem and Dedekind-Cut Axiom are logically equivalent; this remark is due to Veblen. The proof that the Borel Theorem has the Dedekind-Cut Axiom as a consequence, assumes that the interval is divided into the groups $A$ and $B$, as conditioned in the preceding paragraph. With every point of the interval, we associate as far as possible an interval containing the point, and consisting only of points of group $A$ or of group $B$. Either every point of $(a, b)$ is interior to one of these intervals, or the contrary is true. In the first case, we have by the Borel Theorem a finite number of intervals, reaching from $a$ to $b$, containing every point as an interior point. Since $a$ belongs to $A$, the intervals overlap, and each interval contains only points of $A$ or $B$, it follows that every point of $(a, b)$ is a point of $A$, which is contrary to hypothesis. Hence there exists at least one point which is not interior to an interval containing only points of $A$, or of $B$, i. e., satisfies the conditions of the Cut Axiom. An analysis of this proof shows that the theorem is a close relative of the following theorem.

*If a function is not invariant in sign throughout an interval $(a, b)$, then there exists a point of the interval in every vicinity of which the function is not invariant in sign.*†

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* To be a complete analog to mathematical induction, condition (2) should read: If S is true for all points preceding $x'$, then it is true for $x'$. In this form, however, it does not seem to have the power which carries one to the end of the intervals.

† K. P. Williams, ANNALS OF MATHEMATICS, (2), vol. 17 (1915–6), pp. 72–3.

The Induction Principle also furnishes a method of proof for the Lebesgue Chain Theorem. Two have been given along similar lines. The first, due to J. Pal,* considers the case in which to every point of $(a, b)$ there corresponds only a single interval having this point as left-hand end-point. In that case, any chain beginning from $a$ is necessarily unique. The second, due to G. C. Young,† applies to the general case, where the number of intervals associated with a given point is not specified. Obviously in this case there may be many different chains leading from one point to another.

We say that the set $\mathfrak{G}$ of intervals $I$ forms a chain from the point $a$ to the point $x$ in case $\mathfrak{G}$ is a set of non-overlapping intervals such that every point of $(a, x)$, excepting perhaps $x$, is either an interior point or a left-hand end-point of one of the intervals of $\mathfrak{G}$. The point $x$ may be interior to an interval, or an end-point of a chain, i. e., a right-hand end-point of an interval, or the limiting point of a sequence of intervals of the chain $\mathfrak{G}$.

In applying the Induction Principle assume that the point $x$ is such that for every $y$ less than $x$, there exists a chain from $a$ to $y$. The case in which there is an end-point $y$ of a chain from $a$ to $y$, such that there exists an interval with $y$ as left-hand end-point which contains $x$, and the case in which $x$ is an end-point of a chain leading from $a$ to $x$, are easily disposed of. If neither of these two cases are holding, let $e_1, e_2, e_3, \cdots, e_n, \cdots$ be a monotonic decreasing sequence of positive numbers converging to zero. Then by hypothesis there exists a chain from $a$ to $x_1$, where the distance of $x_1$ to $x$ is less than $e_1$. If there exists a chain extending from $x_1$ beyond $x$, then by adding this chain to the chain from $a$ to $x_1$, we extend beyond $x$. In the contrary case, let $y_1$ be the least upper bound of points reached by chains beginning at $x_1$. Then there exists a chain from $x_1$

* Palermo Rendiconti, vol. 33 (1912), pp. 352–3.

† Bulletin des Sciences Mathématiques, (2), vol. 43 (1919), pp. 245–7.

to a point $x_2$ with $y_1 - x_2 < e_2$. By a repetition of this process, we get a sequence of points $x_1, x_2, x_3, \cdots$ having a limiting point $x_0$, and by the method of construction, by combining chains, we get a chain from $a$ to $x_0$. Now $x_0$ is a left-hand end-point of an interval of the family $\mathfrak{F}$ whose length is greater than $e_n$ for $n$ sufficiently large. But this interval added to the chain from $a$ to $x_0$ would give us a chain reaching beyond $y_n$ for $n$ sufficiently large, contrary to the definition of $y_n$. It follows that there is a chain from $x_1$ to $x$, and so beyond $x$.

A careful analysis of the proof shows that its basis is really the same as the proof of this theorem using the Cantor numbers, viz., the fact that there are at most a denumerable set of non-overlapping intervals.

3. *Extensions of the Borel Theorem.* (a) *To Closed Sets.* An almost immediately obvious extension of the Borel Theorem is to replace the closed interval covered by the family $\mathfrak{F}$ by a closed set of points $E$. The methods of proof sketched above all apply excepting that in the case of (e) and (f) which involve order on a line, it may be necessary to take account of the intervals complementary to the closed set. An alternative method of procedure is to take the smallest interval $(a, b)$ containing the set $E$, and enlarge the family $\mathfrak{F}$ by the addition of the intervals belonging to the complement of $E$ with respect to $(a, b)$ and then apply the theorem for the interval.*

(b) *Extension to n-Dimensional Space.*† A further extension which is possible is to $n$-dimensional space. This was conceived almost as soon as the Borel Theorem, due to the consideration of functions of the complex variable.‡

* Cf. also W. H. Young, PROCEEDINGS OF THE LONDON SOCIETY, vol. 35 (1902), pp. 387–8.

† Cf. Schoenfliess-Hahn, *Entwickelungen*, vol. 1, 1913, pp. 239–241.

‡Cf. Cousin above. The Cauchy-Goursat theorem is virtually based on a two-dimensional Borel Theorem. See TRANSACTIONS OF THIS SOCIETY, vol. 1 (1900), pp. 14–16.

We define the point $x$ *interior to a set of points $I$ of $n$-dimensional space* to mean that there exists a vicinity of $x$ (i. e., an $n$-dimensional sphere or cube having $x$ as center) containing only points of $I$. Then the Borel Theorem reads as follows.

*If $E$ is any closed set of $n$-dimensional space covered by a family $\mathfrak{F}$ of sets $I$ (i. e., such that every point of $E$ is interior to some member of the family) then a finite subfamily of $\mathfrak{F}$ covers $E$.*

Some of the proofs given for the linear interval are immediately extensible to space. This is true of the methods (a) and (b) by successive subdivisions, and (c) in which the family $\mathfrak{F}$ is replaced by a family of spheres with the points of $E$ as centers, also of (d) the denumerable case. (e) and (f) seem to use particularly linear order and consequently are effective mainly in proofs by induction, passing from the case of $n$-dimensional to $(n+1)$-dimensional space. Lebesgue* suggests an ingenious method of passing from the plane to the linear interval by using the Peano curve which maps the square on the linear interval.

4. *Necessary Conditions. Lindelöf Theorem.* The hypothesis of the Borel Theorem specifies as sufficient conditions that the set $E$ be closed and bounded. These conditions are also necessary. If $E$ is not closed then there exists a point $x_0$ limiting point of $E$ not belonging to $E$. If we surround each point $x$ of $E$ by a sphere of radius one-half of the distance of $x$ from $x_0$, then no finite subfamily of this family of spheres will cover $E$. The fact that $E$ must be bounded is obvious.

In the same direction is the remark that the Weierstrass-Bolzano Theorem is a consequence of the Borel Theorem. For suppose an infinite set $E_0$ of points $x_1, \cdots, x_n, \cdots$ contained in a bounded closed region $E$. Then either for every point $x$ of $E$ there exists a sphere around $x$ containing at most one point of $E_0$, or there exists a point $x$ such that

* *Leçons*, p. 119.

every sphere about $x$ contains an infinity of points selected from $E_0$. In the first case, the spheres constitute a family $\mathfrak{F}$ for $E$ to which the Borel Theorem applies. But a finite number of spheres of $\mathfrak{F}$ contain only a finite number of points of $E_0$. Hence $E_0$ has a limiting point in $E$.

We note in passing that it is immaterial whether the family $\mathfrak{F}$ consist of open sets (containing only interior points) or arbitrary sets of points to make the Borel Theorem valid.

Returning to the linear case, the fact that the Borel Theorem does not hold for sets in general suggests the question what can be said about the set of points interior to any family of intervals. If by the term *two families of intervals are equivalent* we mean that they cover the same set of points, then we have on the one hand

*Any family of intervals is equivalent to a family of non-overlapping intervals.*

This is immediately evident. We need only take any point interior to some interval and proceed to the left and right until we meet a point which is not interior to any interval of the given family. In this way we define a group of non-overlapping sets which is denumerable.

As an analog to the Borel Theorem we have the following theorem, which is due to Lindelöf.*

LINDELÖF THEOREM. *In any family of intervals it is possible to find a denumerable sub-family having the same interior points.*

This theorem has been proved in different ways. Mention might be made of the following proofs:

(a) *Using Density of Rational Points on a Line.*† Let $x$ be any point interior to some interval $I$ of the family. Then there exists an interval $R$ with rational end-points containing $x$, and entirely interior to $I$. This sets up a correspondence between a family of the given intervals and the

* COMPTES RENDUS, vol. 137 (1903), p. 697.

† Cf. W. H. Young, PALERMO RENDICONTI, vol. 21 (1906,) p. 125.

family of intervals with rational end-points, which is denumerable.

We observe that this method of proof makes use of the density of the rational points on a line and their denumerability. An equivalent method of constructing the intervals with rational end-points, would be to use the rational points interior to the family of intervals, and make each of them the mid-point of an interval with rational end-points interior to some interval of the set. In this form, the proof of the corresponding theorem in $n$-space can be made.

(b) *Via the Borel Theorem.* Let $(a, b)$ be one of the intervals of the equivalent family of non-overlapping intervals, and $x_0$ any point of $(a, b)$. Let $x_0, x_{-1}, x_{-2}, \cdots$ be a monotonic sequence of points approaching $a$, and $x_0, x_1, x_2, \cdots$ be a monotonic sequence approaching $b$. Then the closed intervals $(x_{-n}, x_n)$ can be covered by a finite number of intervals of the given family $\mathfrak{F}$. This gives us a method for selecting a denumerable set of intervals having the desired property relative to $(a, b)$. The final result is a consequence of the properties of denumerability.

(c) *Lindelöf Proof.* Suppose the points covered by $\mathfrak{F}$ belong to a finite interval $(a, b)$. For any $x$ covered by $\mathfrak{F}$, let $d_x$ be the maximum of the values of $d$ for which $(x-d, x+d)$ is interior to one of the intervals of the family $\mathfrak{F}$, and consider the set $E_n$ of points of the set $E$ covered by $\mathfrak{F}$, for which $d_x > 1/n$. Then by the observation of §2(c), it follows that the set $E_n$ can be covered by a finite number of intervals chosen from the family $\mathfrak{F}$. This gives a method for the enumeration of the subfamily. The extension to the unbounded interval results from the fact that it can be divided into a denumerable set of finite intervals.*

A similar method of procedure applies in $n$-space.

* W. H. Young, PALERMO RENDICONTI, vol. 21 (1906), pp. 126–7, gives another proof utilizing somewhat similar procedure, which resulted in a series of polemics with Schoenflies, the final shot being fired by the latter in PALERMO RENDICONTI, vol. 35 (1913), pp. 74–78. At best the method of proof, even as validated by Schoenflies, is not satisfactory.

5. *Strict Families of Intervals.** It is obvious that in general the selection of the finite subfamily of intervals in the Borel Theorem may be made in many different ways. Obviously too, some of the intervals of the finite family may be unnecessary in that all points interior to them are interior to other intervals of the family. We shall call a *strict* family of intervals, a family in which each interval is necessary, in the sense that it contains a point not interior to any of the other members of the family. We have then

*Any finite family of intervals can be replaced by an equivalent strict family.*

We note first that if three intervals have a common point they can be replaced by at most two of these intervals. Let the intervals be $(a_1, b_1)$, $(a_2, b_2)$, and $(a_3, b_3)$, where since they contain a common interior point, the notation is chosen so that

$$a_1 \leqq a_2 \leqq a_3 \leqq b_1 .$$

Then obviously if $b_2 < b_3$, we can dispense with $(a_2, b_2)$ and if $b_2 > b_3$, then we can dispense with $(a_3, b_3)$. By the use of this result the theorem stated is immediate. The process of deletion in any particular case may be tedious, especially if governed by other considerations, such as, for instance, the desire to make the sum of the lengths of the retained intervals a minimum.

More generally we have the following theorem.

*If any family of intervals is such that every point interior to one of the intervals is interior to at most a finite number of intervals of the family, then we can select a strict subfamily, equivalent to the given family.*

We note in the first place that the given family is necessarily denumerable. For to each point interior to an interval of the family there will correspond a vicinity which is common to a definite finite number $n_x$ of intervals. By

* Cf. Denjoy, JOURNAL DE MATHÉMATIQUES, (7), vol. 1 (1915), pp. 223–30.

the Lindelöf Theorem these vicinities can be replaced by a denumerable set of the same vicinities, and since to each of the final vicinities there corresponds a finite number $n_x$ it follows that the original family of intervals is denumerable.

Let then the intervals be arranged in sequential array $I_1, \cdots, I_n, \cdots$. Let $I_{n_1}$ be the first interval of the array covering a point not interior to any of the succeeding intervals. Such an interval will always exist. We determine $I_{n_m}$ as the first interval of the sequence following $I_{n_{m-1}}$ containing a point not interior to $I_{n_1}, \cdots, I_{n_{m-1}}$, and a point not interior to any of the intervals following $I_{n_m}$. It is obvious that the resulting family is a strict family. It remains to show that every point $x$ covered by the original family is covered by the subfamily. Since $x$ is an interior point of a finite number of intervals of $\mathfrak{F}$ the indices of the sequence $I_1, \cdots, I_n$ to which $x$ is interior have a definite maximum $N$, i. e. there will be an index $n_m < N$ such that $x$ is interior to $I_{n_m}$.

From the point of view of measure, a strict family of intervals has the property that the sum of the lengths of the intervals is less than double the sum of the lengths of the intervals covered. For consider any strict family of intervals. We note first of all that it is necessarily denumerable. For since each interval contains a point not interior to other intervals it contains a subinterval having no points in common with other intervals. These subintervals define a system of non-overlapping intervals which is denumerable, and consequently the original family, which is in one-to-one correspondence with it, is denumerable. Let $x_n$ be a point interior only to $I_n$. Then the points $x_n$ have as limiting points only points not interior to any interval of the family. From the contrary assumption would follow that some of the points $x_n$ are interior to more than one interval of the given family. As a consequence the points $x_n$ in each interval $(a_k, b_k)$ of the equivalent family of non-overlapping intervals can be arranged in order so that between two

points no further points of the sequence appear. Consequently no point of $(a_k, b_k)$ appears in more than two intervals of the family.

Denjoy has given a condition under which the reduction of a family to an equivalent strict subfamily is possible. If a family $\mathfrak{F}$ of intervals is *upper semi-closed* in case every interval which is the limit of a sequence of intervals $I_n$ chosen from the family is part of an interval of or belongs to $\mathfrak{F}$, then we have the following theorem.

*From every upper semi-closed family $\mathfrak{F}$ of intervals it is possible to select an equivalent strict subfamily.*

It will obviously be sufficient to show the possibility of selecting an equivalent subfamily $\mathfrak{F}_0$, such that every point covered by $\mathfrak{F}$ is interior to at most a finite number of intervals of $\mathfrak{F}_0$. Let $(a, b)$ be an interval of the equivalent family $\mathfrak{G}$ of non-overlapping intervals. There are three possibilities:

(a) The points $a$ and $b$ are both end-points of intervals of $\mathfrak{F}$. Then, applying the Borel Theorem, we get a finite sub-family having every point of $(a, b)$ excepting $a$ and $b$ as interior points.

(b) Both $a$ and $b$ are not end-points of intervals of $\mathfrak{F}$. Then by successive applications of the Borel Theorem, as in the proof of the Lindelöf Theorem (§3(b)), we construct an equivalent family which has the property that $a$ or $b$ is a limiting point of end-points of any infinite set of intervals selected from this family. If possible let $x$ be an interior point of $(a, b)$ interior to an infinite number of intervals of the resulting subfamily. Then $a$ or $b$ is a limiting point of end-points of these intervals. By the semiclosure of the family $\mathfrak{F}$ it follows that $a$ or $b$ is then an end-point of an interval reaching at least to $x$, contrary to the assumption. Hence every point interior to $(a, b)$ is interior to at most a finite number of intervals of this subfamily.

(c) If only one of the end-points $a$ or $b$ is an end-point of an interval of $\mathfrak{F}$, then a process similar to case (b) utilizing only one end-point will apply.

As a corollary of this result we have the following theorem.*

*If a family $\mathfrak{F}$ of intervals is such that for every e there are at most a finite number of intervals of $\mathfrak{F}$ of length greater than e, then there exists an equivalent strict subfamily of $\mathfrak{F}$.*

6. *Other Theorems on Reduction of Families of Intervals to Finite Subfamilies.* We turn our attention briefly to a number of analogs of the Borel Theorem, most of which are due to W. H. and G. C. Young.† A theorem which has the Borel Theorem as a consequence, but for which the converse has not yet been shown, has been called by the Youngs the Heine-Young theorem, because of its similarity to Heine's proof of the uniform continuity theorem.

HEINE-YOUNG THEOREM. *With every point x of a closed interval (a, b) there are associated two intervals, an $R_x$ having x as left-hand end-point, and $L_x$ having x as right-hand end-point. These intervals are connected by the condition that if $x'$ is interior to the $L_x$ for x, then $R_{x'}$ contains x as an interior point or an end-point. Then a finite number of the R intervals cover (a, b) without overlapping.*

The $R$ intervals without the intervention of the $L$ intervals are equivalent to a Lebesgue chain, and since the $R_x$ is unique for every point there will be only one such chain. The presence of the $L$ intervals insures the finiteness of the chain by preventing limiting points. For the $L$ corresponding to a possible limiting point $x$ would include an infinite number of $x'$ whose $R_{x'}$ by hypothesis should reach up to or beyond $x$.

The finiteness of the $R$ chain is also apparent from another point of view. The $L$ intervals are equivalent to a chain. Now any $L$ is covered by at most two $R$ intervals. We can then as in the case of the Borel Theorem define a sequence

* Cf. R. L. Moore, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 10 (1924), 466–7.

† Cf. *Reduction of intervals*, PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 14 (1915), pp. 111–130.

of decreasing ordinals by beginning with $R_a$. For if a point $x'$ is a limiting point of the $L$ intervals, of the chain, the corresponding $R_{x'}$ will cover an infinite number of the $L$ intervals, and have as right-hand end-point a point interior to an $L$ or an end-point of an $L$ or a point which is again a limiting point. In either of these cases we can proceed in the formation of our decreasing ordinal series, which is finite.

While this theorem seems to be more general than the Borel Theorem, its usefulness is rather hampered by the peculiar way in which the two sets of intervals are interlaced.

Another theorem of the same type is due to Lusin*, which is stated by him as applying to non-dense perfect sets.

LUSIN THEOREM. *Let $E$ be any non-dense perfect set, and $\mathfrak{G}$ the family of open intervals $(a_k, b_k)$ complementary to $E$ relative to an enclosing interval $(a, b)$. Moreover, suppose that to every point $x$ of $E$ not an $a_k$ or $b_k$ there correspond intervals of a family $\mathfrak{F}$ to the right of $x$ of the form $(x, a_k)$, and all intervals to the left of the form $(b_k, x)$ in a certain neighborhood of $x$; while to the points $b_k$, only intervals of first type correspond, and to points $a_k$ only intervals of the second type; then a finite number of the intervals from $\mathfrak{F}$ and $\mathfrak{G}$ cover $(a, b)$ or $E$ without overlapping.*

The proof can be made along the lines of the Young Theorem; the right-hand intervals together with the intervals of $\mathfrak{G}$ can be formed into a chain, this being reduced to a finite set by the inclusion of the left-hand intervals. From this it is apparent that to prevent overlapping it is not sufficient to assume as Lusin does that only a single interval the right and left corresponds to each point of E.

W. H. and G. C. Young credit the following theorem for a closed interval to Lusin.

*If to every point $x$ of a closed interval (or closed point set) there correspond all the intervals to the left and right of the*

* MOSCOW MATHEMATICAL SOCIETY TRANSACTIONS (SBORNIK), vol. 28 (1911–12), p. 270.

*point in a given neighborhood of the point, then a finite number of these intervals suffice to cover the interval (or point set) without overlapping.**

With every $x$ of $(a, b)$ we associate an interval of a new family such that $x$ is the middle point of its interval, and the interval consists of an $L$ and an $R$ of equal length selected from intervals associated with the point. A finite number of these intervals suffice to cover $(a, b)$. We assume that the intervals retained give a strict covering for $(a, b)$. Then if $I^*_{n_1}$ contains $a$, let $x_1$ be its middle point. If $x_1 > a$, we reach $x_1$ by noting that $(a, x_1)$ belongs to the given family. Let $I_{n_2}$ overlap with $I_{n_1}$ and $x_2$ be its middle point. Then $x_2 > x_1$. If $x_2$ belongs to $I_{n_1}$ then $x_1$ and $x_2$ is an interval of the given family. If $x_2$ is not in $I_{n_1}$ then we reach $x_2$ by taking the intervals $x_1\ b_1$ and $b_1\ x_2$, where $b_1$ is the right-hand end-point of $I_n$. It is obvious that in this way we can construct a finite number of intervals as required.

A slightly more general theorem can be obtained by associating with every point all the intervals to the left in a certain vicinity, and only one or more to the right. In this form the proof given above via the Borel Theorem is not valid, and it is not clear whether the Borel Theorem can be used. It can be deduced by using the Lebesgue chain associated with the intervals to the right; or, following the Youngs, we get an equivalent method by associating with every point $x$ of $(a, b)$ as an $R_x$ the smallest interval containing all the intervals of the given family having $x$ as left-hand end-point and as $L_x$ the given vicinity to the left. Then the Heine-Young Theorem applies; there exists a finite number of the $R$ intervals reaching from $a$ to $b$. Any of these $R$ intervals can be replaced by at most two intervals of the original family.

* This theorem can be made the basis of a proof of the theorem *If $f(x)$ has a Riemann integrable derivative $f'(x)$ on $(a,b)$ then $f(b)-f(a)=\int_a^b f(x)dx$*, without using the mean-value theorem of the differential calculus, a desideratum in considerations in general functional space.

In trying to avoid the infinite elements which pervade the Lebesgue Chain Theorem, the Youngs state the following lemma.

YOUNG'S LEMMA. *If to every point of the closed interval $(a, b)$ there corresponds an interval of a family $\mathfrak{F}$ having this point as left-hand end-point, then for every $e$, there exists a finite subfamily $\mathfrak{F}_e$ of non-overlapping intervals, such that he sum of the complementary intervals is less than $e$.*

If we assume the Lebesgue Chain Theorem this result is immediate. The Youngs prove the theorem by an ingenious application of the Heine-Young Theorem. Chose $n$ so that

$$2(b-a)<ne.$$

Then to every point $x$ of $(a, b)$ we make correspond as $L_x$ an interval to the left of $x$ whose length is $1/n$ of the least upper bound of the intervals of $\mathfrak{F}$ associated with $x$. If $(x, x+h_x)$ is any interval of $\mathfrak{F}$, then we add to $\mathfrak{F}$ all the intervals $(y, x+h_x)$ where

$$x-\frac{h_x}{n}\leqq y\leqq x.$$

Let $R_x$ be the smallest interval containing all the intervals of this extended family which have $x$ as left-hand end-point. Then it is apparent that the $R$ and $L$ intervals satisfy the conditions of the Heine-Young Theorem, so that a finite number of the $R$ intervals extend from $a$ to $b$ without overlapping. The result desired follows from the fact that each $R$ can be approximated up to $2/n$ of its length by an interval of the family $\mathfrak{F}$.

It is obvious that the Young Lemma will hold also for closed sets of points $E$, in which form it was originally stated.* A proof similar to this other proof given by Young has been made by Sierpinski† for the following more general theorem which utilizes the properties of upper measure.

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* PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 9 (1911), pp. 325ff.

† Cf. FUNDAMENTA MATHEMATICAE, vol. 4 (1923), pp. 201–3.

*If $E$ is any bounded linear set, and $\mathfrak{F}$ a family of intervals such that to every $x$ of $E$ there corresponds an interval $R_x$ of $\mathfrak{F}$, having $x$ as left-hand end-point, then for every $e$ it is possible to determine a finite subfamily $\mathfrak{F}_e$ of $\mathfrak{F}$, consisting of non-overlapping intervals, and such that*

$$\overline{m}(E - E \cdot F_e) < e.^*$$

The proof is as follows. Let $E_n$ be the set of points of $E$ for which there exists an $R_x$ of length greater than $1/n$. Then the $E_n$ form a monotonic increasing family of sets such that

$$E = \sum E_n,$$

and consequently

$$\lim_n \bar{m}E_n = \bar{m}E.$$

Chose $n$ so that

$$\bar{m}E - \bar{m}E_n < \frac{e}{2}.$$

Let $(a_1, b_1)$ be the smallest interval containing $E_n$, its length being $l$. Obviously if we use intervals of $\mathfrak{F}$ of length greater than $1/n$, then there will be at most $nl$ possible non-overlapping intervals in $(a_1, b_1)$. If between every two intervals we allow a space $d$ such that

$$nld < \frac{e}{2},$$

then the points of $E_n$ not covered will be of upper measure at most $e/2$. Since $a_1$ is a lower bound of points of $E_n$ there will be a point $x_1$ of $E_n$ in the interval $(a_1, a_1+d)$ and an $R_x$ of length greater than $1/n$. Let $a_2$ be the lower bound of the points of $E_n$ to the right of $R_{x_1} = R_1$. Take $x_2$ so that $x_2$ belongs to $(a_2, a_2+d)$ and $E_n$. Continuing in this manner we get a finite number of intervals $R_1, \cdots, R_m$. Let $S$ be the set of points belonging to $R_1, \cdots, R_m$. Then we wish to show that

$$(\bar{m}E - ES) < e.$$

* We follow the usual notations: $\bar{m}E$ or $\overline{\text{meas}}\ E$ for upper measure of $E$; $E \cdot F$ the set of points common to $E$ and $F$; $E+F$ set of points belonging to either $E$ or $F$; $E-F$ the sets of points of $E$ not in $F$.

Now by the properties of upper measure we have*

$$\bar{m}E = \bar{m}(E - ES) + \bar{m}ES.$$

By the selection of $E_n$, we have

$$\bar{m}E - \bar{m}E_n < \frac{e}{2},$$

and by the selection of $S$,

$$\bar{m}E_n - \bar{m}E_nS < \frac{e}{2},$$

so that

$$\bar{m}E - \bar{m}E_nS < e.$$

But $E$ contains $E_n$ and so $ES$ contains $E_nS$. Hence

$$\bar{m}(E - ES) = \bar{m}E - \bar{m}ES < \bar{m}E - \bar{m}E_nS < e.$$

These theorems are of value in connection with the discussion of the distribution of infinite derivatives of a function of a single variable. The Young Lemma and the Sierpinski extension have been used by the Youngs† and by Rajchman and Saks‡ to obtain in a simple way forms of the theorem that any monotonic function and therefore any function of bounded variation has a finite derivative excepting at a set of points of measure zero.

Closely related to these theorems is a so-called "tile" theorem of the Youngs. If to a point correspond all the intervals in a certain vicinity of the point, then any such interval is called a *tile* and the point is called *the point of attachment.*

TILE THEOREM.§ *Suppose a family of intervals* $\mathfrak{F}$ *such that to every point* $x$ *of a linear bounded set* $E$*, there correspond all the intervals in a certain neighborhood of* $x$*. Then for*

* Cf. Hausdorff, *Mengenlehre*, p. 415.

† PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 9 (1911), pp. 325–35.

‡ FUNDAMENTA MATHEMATICAE, vol. 4 (1923), pp. 204–13.

§ PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 2 (1904), pp. 67–9; ibid., (2) vol. 14 (1915), pp. 122–126. The Youngs assume that $E$ is measurable.

*every $e$ and $d$, there exists a finite or denumerably infinite subfamily $\mathfrak{F}_{de}$ of $\mathfrak{F}$, such that*

(*a*) *every interval $I$ of $\mathfrak{F}_{de}$ is of length less than $d$,*

(*b*) *the point associated with each interval of $\mathfrak{F}_{de}$ is interior only to that interval,*

(*c*) *every point of $E$ is interior to some interval of $\mathfrak{F}_{de}$, and*

$$(d) \qquad \sum mI_n - \bar{m}E < e$$

*where $I_n$ are the intervals of $\mathfrak{F}_{de}$.*

In case the set $E$ is a closed interval, the set $\mathfrak{F}_{de}$ is finite.* In this case the proof can be made via the Borel Theorem. We discard all intervals of $\mathfrak{F}$ of length greater than $d$, and then associate with each point $x$ an interval of the remaining family associated with $x$, of which $x$ is the middle point. Then replace this new family by a finite strict subfamily via the Borel Theorem. Let $x_1, \cdots, x_n$ be the mid-points of the resulting intervals arranged in order. We can then select intervals from $\mathfrak{F}$ attached to the points $x_1, \cdots, x_n$ in such a way that the overlap lies entirely between $x_i$, $x_{i+1}$ and has for each $i$ a length less than $e/n$.

In the case of any set $E$, we enclose $E$ in a set of non-overlapping intervals $J_n$ such that

$$\sum mJ_n - \bar{m}E < e.$$

Discard all the intervals of $\mathfrak{F}$ of length greater than $d$, and those which do not lie completely interior to some $J_n$. Associate with each point $x$ of $E$ an interval of the remaining intervals associated with $x$, having $x$ as middle point, forming a family $\mathfrak{F}_0$. The family $\mathfrak{F}_0$ is equivalent to a denumerable family of non-overlapping intervals $K_m$ lying interior to the $J_n$, which contains $E$, so that

$$\sum mK_m - \bar{m}E < e.$$

* Also true if $E$ is a closed set. Obviously this case is closely related to the Lusin-Young Theorem.

Now by following a process similar to that used in §5, and by discarding, at each stage, intervals not needed, we can reduce $\mathfrak{F}_0$ to an equivalent strict family $\mathfrak{F}$ of intervals. This strict family of intervals can be replaced by a family $\mathfrak{F}_{d_0}$ of intervals chosen from $\mathfrak{F}$, satisfying the conditions $(b)$ and $(d)$, by a line of reasoning similar to that used in connection with the case where $E$ is a closed interval.

7. *The Vitali Theorem.* The tile theorem of the Youngs is very closely related to the Vitali Theorem, which plays a role in connection with measurable sets and derivatives of functions comparable to that of the Borel Theorem relative to closed sets and continuous functions. It was stated by Vitali* for the linear interval in a form equivalent to the following:

*If $\mathfrak{F}$ is a family of intervals such that for every point $x$ of the bounded measurable set $E$, there exists a set of intervals of $\mathfrak{F}$ containing $x$, whose lengths approach zero, then there exists a denumerable subfamily of $\mathfrak{F}$ consisting of non-overlapping intervals, the sum of whose lengths is greater than the measure of $E$.*

This theorem has been extended in various ways, especially to higher dimensions, the chief extensions being due to Lebesgue† and Carathéodory.‡ Probably the simplest statement and proof of the fundamental extended form of the theorem has been given by Banach,§ viz.

VITALI THEOREM. *Let $E$ be any bounded set of points in a space of $n$ dimensions. Let $\mathfrak{F}$ be a family of closed sets of points $I$, such that to each point $x$ of $E$, there corresponds a sequence $I_n$ chosen from $\mathfrak{F}$ subject to two conditions:*

(1) *if $r_n(x)$ is the radius of the smallest sphere $C_n(x)$ of center $x$ containing $I_n(x)$, then $\lim_n r_n(x) = 0$, and*

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* ATTI DI TORINO, vol. 43 (1907), pp. 229–236.

† ANNALES DE L'ÉCOLE NORMALE, (3), vol. 27 (1910), pp. 391–5.

‡ *Vorlesungen über reelle Funktionen*, 1918, pp. 299–307.

§ *Sur le théorème de Vitali*, FUNDAMENTA MATHEMATICAE, vol. 5 (1924), pp. 130–6.

(2) *there exists a quantity $\alpha$ independent of $n$ and $x$ such that*

$$\frac{\text{meas } I_n(x)}{\text{meas } C_n(x)} > \alpha.$$

*Then for every $e$, there exists a finite or denumerable family of sets $I_n(x_n) = I_n'$ (1) without common points, and such that*

(2) $$\sum \text{meas } I_n' < \overline{\text{meas}}\, E + e$$

*and*

(3) $$\overline{\text{meas}}\,(E - \sum I_n' E) < e.$$

In this theorem it is immaterial whether the sets $I_n(x)$ contain $x$.* Also it is possible to replace the spheres $C_n(x)$ by cubes having $x$ as center, or rectangular parallelepipeds, the ratios of whose dimensions are bounded from infinity and zero. That the theorem is not true with an unconditioned set of rectangular parallelepipeds has been shown by Banach.†

We give the proof in two dimensions, the changes to be made for the $n$-dimensional case being obvious.

Since $E$ is bounded, it is possible to find an open set $U$ containing all the points of $E$ and such that

$$\text{meas } U \leqq \overline{\text{meas}}\, E + e.$$

We then reduce the family $\mathfrak{F}$ to the family $\mathfrak{F}_1$, by retaining only those $I_n(x)$ which are contained in $U$, so that condition (2) of the conclusion is fulfilled if the selection can be made in accordance with condition (1).

Let $k$ be any constant greater than unity. Then by selecting an

$$I'_1 = I_{n_1}(x_1),$$

such that the radius $r_{n_1}(x_1)$ is sufficiently near to the least upper bound of the radii $r_n(x)$ we can make sure that

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* In Lebesgue's formulation (loc. cit., p. 391) the $I_n(x)$ contain $x$ but then the spheres containing $I_n(x)$ need not have $x$ as center.

† Loc. cit., pp. 134–6.

$$r_n(x) < kr_{n_1}(x_1) = kr(I_1'),$$

for every $n$ and $x$. Similarly if $I_1', \cdots, I_{m-1}'$ have been determined, then by a similar method we can select the set

$$I_m' = I_{n_m}(x_m)$$

with properties (*a*) $I_m'$ does not have any points in common with $I_j'$ for $j \leqq m-1$, and (*b*)

$$r_n(x) < kr_{n_m}(x_m) = kr(I_m')$$

for all $I_n(x)$ which have no points in common with $I_j'$ for $j \leqq m-1$. Since $E$ and therefore also $U$ is bounded, and consequently has finite upper measure, it follows that meas $I_m'$ approaches zero with $m$ and consequently $r(I_m')$ approaches zero with $m$, due to condition (2) of the hypothesis. Let $C_m'$ be the circle of radius $(2k+1)\,r(I_m')$, center $x_m$. Then we show that for every $m$, all points of $E$ belong to

$$\sum_1^{m-1} I_i' + \sum_m^{\infty} C_j'.$$

We observe first that the sets $I_n'$ for $n \geqq m$ are chosen from the reduced family $\mathfrak{F}_m$ of $\mathfrak{F}$ with respect to the open set

$$U_m = U - \sum_1^{m-1} I_j'$$

as $I_n'$, for $n \geqq 1$, was selected from the reduced family $\mathfrak{F}_1$ with respect to $U$. Let $x_0$ be any point of $E$. Then either $x_0$ belongs to some $I_j'$ or it belongs to $U_m$ for every $m$. Let $I_m(x_0)$ be the $I$ of maximum radius contained in $U_m$. Then since $r(I_n)$ approaches zero with $n$, there exists a value of $n$ such that

$$r(I_m(x_0)) > kr(I_n'),$$

but

$$r(I_m(x_0)) \leqq kr(I_j'),$$

for $j = m, \cdots, n-1$. It follows that $I_m(x)$ must have parts in common with one of the sets $I_m', \cdots, I_{n-1}'$, and hence

is completely covered by the corresponding $C_i'$ by the method of formation of the $C_i'$.

Now since

$$\sum_1^\infty \text{meas } I_i' < \overline{\text{meas}}\, E$$

it follows that $\sum_m^\infty$ meas $I_i'$ approaches zero with $m$. Hence since

$$\sum_m^\infty \text{meas } I_i' > \alpha \sum_m^\infty \text{meas } C_i' = \frac{\alpha}{(2k+1)^2} \sum_m^\infty \text{meas } C_i'$$

it follows that $\sum_m^\infty$ meas $C_i'$ approaches zero with $m$, i.e., the set $\sum_1^\infty I_i'$ contains all points of $E$ up to a set of measure zero.

The following generalization is possible.*

*The Vitali Theorem is still valid in case the set $E$ is any set, and the constant $\alpha$ of condition* (2) *of the hypothesis is dependent on $x$, but independent of $n$.*

The case when $E$ is any set can be reduced to the case when $E$ is bounded, by consideration of the fact that $n$-dimensional space can be broken up into a denumerable set of compartments of finite magnitude. For the case where $E$ is bounded and $\alpha$ is dependent on the point $x$, we consider the sets $E_n$ of points of $E$ which satisfy the conditions

$$\frac{1}{n+1} < \alpha(x) \leqq \frac{1}{n}$$

where $\alpha(x)$ is the greatest lower bound of the values of the fraction

$$\frac{\text{meas } I_m(x)}{\text{meas } C_m(x)}.$$

We can then apply the previous theorem to the sets $E_n$, successively. We determine $I_1', \cdots, I_n'$ so that

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* Lebesgue, loc. cit., p. 393; Carathéodory, loc. cit., p. 305.

$$\text{meas}\left(E_1 - \sum_{i=1}^{n} E_1 I_i'\right) < \frac{e}{2},$$

then $I'_{n+1}, \cdots, I_{n_1}$ belonging to $U - \sum_{i=1}^{n_1} I_i'$, so that

$$\text{meas}\left(E_1 + E_2 - \sum_{i=1}^{n_2} (E_1 + E_2) I_i'\right) < e\left(1 - \frac{1}{2^2}\right),$$

and so on.

The Vitali Theorem gives a very elegant method for demonstrating the following generalization of the Lebesgue metric density theorem. For the statement of the theorem we define the upper metric density of a set $E$ at a point $x$ of $E$ as

$$\lim_{r \to 0} \frac{\overline{\text{meas}}\, C_r E}{\text{meas}\, C_r}$$

where $C_r$ is a sphere of radius $r$ and center $x$. Then the theorem is*

*The points of any set $E$ at which the metric density is not unity form a set of zero measure.*

For let $E_0$ be the set of points of $E$ at which

$$\lim_{r \to 0} \frac{\overline{\text{meas}}\, C_r E}{\text{meas}\, C_r}$$

does not exist or is less than unity, and let $E_k$, $k$ a positive integer, be the points of $E$ for which there exists a sequence of circles with radii $r_n$ converging to zero, such that

$$\overline{\text{meas}}\, C_{r_n} E < \left(1 - \frac{1}{k}\right) \text{meas}\, C_{r_n}.$$

Then $E_0 = \sum E_k$. The spheres $C_{r_n}$ form a family $\mathfrak{F}$ as required by the Vitali theorem for the points of $E_k$. Hence for every $e$ there exists a denumerable set of the $C_{r_n}$, $C_m'$ satisfying the conditions

* See Lebesgue, loc. cit., p. 407; H. Blumberg, TRANSACTIONS OF THIS SOCIETY, vol. 24 (1923), pp. 122ff; and W. Sierpinski, FUNDAMENTA MATHEMATICAE, vol. 4 (1923), pp. 167–171, where other references are to be found.

$$\overline{\text{meas}}\, E_k = \overline{\text{meas}} \sum_m C_m' E_k = \sum_m \overline{\text{meas}}\, C_m' E_k,$$

and

$$\sum_m \text{meas}\, C_m' < \overline{\text{meas}}\, E_k + e.$$

But since $E_k$ is part of $E$, we have

$$\overline{\text{meas}}\, C_m' E_k < \overline{\text{meas}}\, C_m' E < \left(1 - \frac{1}{k}\right) \text{meas}\, C_m'.$$

Then

$$\overline{\text{meas}}\, E_k = \sum_m \overline{\text{meas}}\, C_m' E_k < \left(1 - \frac{1}{k}\right) \sum_m \text{meas}\, C_m' < \left(1 - \frac{1}{k}\right)\left(\overline{\text{meas}}\, E_k + e\right),$$

from which

$$\overline{\text{meas}}\, E_k < (k-1)e,$$

i.e. $E_k$ is of measure zero for every $k$, giving the same result for $E_0$.

The question naturally arises what it is possible to do in the matter of selecting the sets $I_n$ so as to contain all points of $E$. H. Rademacher* has given the following which might be considered a generalization of the Young Tile Theorem:

*If with every point $x$ of $E$, there is associated a sequence of spheres, whose radii converge to zero, then for every $e$ it is possible to find a denumerable set of these spheres $E_n$, such that*

$$\sum \text{meas}\, C_n < \overline{\text{meas}}\, E + e$$

*and every point of $E$ is interior to at least one of the spheres.*

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* Monatshefte für Mathematik und Physik, vol. 27 (1916), pp. 189–190.

This is a consequence of the Vitali Theorem and the result that under the hypothesis of the theorem, there exists a denumerable set of spheres covering $E$ and a constant $k$ (which depends upon the dimension of the space) such that

$$\sum \text{meas } C_n < k(\overline{\text{meas}}\ E + e).$$

Essentially the point is that under the given hypothesis it is possible for every $e$ to cover a set of measure zero by a denumerable set of the given spheres, the sum of whose measures is less than the given $e$.

It is obvious that in the Rademacher theorem, the spheres can be replaced by cubes having the points $x$ as center. To further extensions there are limitations. J. Splaya-Neumann* has shown that it is necessary that the points $x$ be the centers of the spheres by giving an example of a plane closed set of measure zero, such that the sum of the measures of the covering circles is always greater than or equal to unity.†

For the linear interval, J. C. Burkill‡ has given an exact covering theorem based on intervals, an extension of the Vitali Theorem. By observing that as a result of the metric density theorem the finite number of intervals of the Vitali Theorem can be chosen so that each of the complementary intervals contains a point of the given set, and combining this with the Borel Theorem he obtains the following slightly complicated result:

*If $\mathfrak{F}$ is a given family of intervals $I$ satisfying the hypotheses of the Vitali Theorem relative to a closed set $E$, and $\mathfrak{G}$ a family of intervals $J$ such that for every $x$ of $E$ all intervals in a certain vicinity of $x$ belong to $\mathfrak{G}$, then there exists an interval $K$ which*

* FUNDAMENTA MATHEMATICAE, vol. 5 (1924), pp. 329–30.

† Cf. also K. Menger, WIENER BERICHTE, vol. 133 IIa (1924), pp. 425–7; and R. L. Moore, loc. cit., pp. 464–5, where examples based on intervals, rectangles, and squares are given, which are not centered relative to the points of association. Moore's example, however, is not with respect to a set of measure zero, as he claims.

‡ FUNDAMENTA MATHEMATICAE, vol. 5 (1924), pp. 322–4.

*for every e is completely covered without overlapping by a finite number of intervals from* $\mathfrak{F}$ *and* $\mathfrak{G}$ *fulfilling the conditions*

$$\text{meas}(\Sigma I_n - (\Sigma I_n)EK) < e, \text{ and } \text{meas}(EK - (\Sigma I_n)EK) < e.$$

We note finally that in the Vitali Theorem, the sets $I$ of the family $\mathfrak{F}$ may be replaced by measurable sets, but then the condition that the subfamily consist of non-overlapping sets must be dropped.*

## II. The Borel Theorem in General Spaces.

Probably no theorem of analysis has contributed more towards the analysis of general spaces than the Borel Theorem. The attempts to derive the theorem in increasingly general situations has led to interesting new properties and characterizations of spaces.

8. *Metric Space.* The first and simplest general space to which the Borel Theorem was extended is now generally called a metric space. The definition of the space and the proof of the theorem in this space were made by Fréchet in his Paris thesis.† A metric space $\mathfrak{D}$ consists of a set of general elements $x$. It is postulated that for every pair of elements $x_1$ and $x_2$ of the space there exists a positive real number $\delta(x_1, x_2)$ called distance and subject to the conditions

(1) $\delta(x_1, x_2) = \delta(x_2, x_1)$ for every $x_1$ and $x_2$,

(2) $\delta(x_1, x_2) = 0$ if and only if $x_1$ and $x_2$ are identical,

(3) $\delta(x_1, x_2) \leqq \delta(x_1, x_3) + \delta(x_3, x_2)$ for every $x_1$, $x_2$ and $x_3$.

A sequence $\{x_n\}$ is said to have as *limit* the element $x$ if $\lim_n \delta(x_n, x) = 0$. A set $E$ in the space $\mathfrak{D}$ is said to have $x$ as *limiting element* if there exists a sequence of distinct elements $\{x_n\}$ extracted from $E$ having $x$ as limit. *Derived sets*, *closed sets*, and *perfect sets* are defined in the usual way,

* See Lebesgue, loc. cit., p. 394; and Rademacher, loc. cit., pp. 191–2.

† *Sur quelques points du calcul fonctionnel*, Palermo Rendiconti vol. 22 (1906), pp. 1–72. Fréchet's results were stated with respect to what seemed to be a slightly more general situation, the equivalence with the above metric space being shown by E. W. Chittenden, Transactions of this Society, vol. 18 (1917), pp. 161–6.

and the notation $E'$ is used for the derived set of $E$. In such a space it seems natural to define a *sphere* center $x_0$ and radius $r$ as the totality of points of $\mathfrak{D}$ satisfying the condition

$$\delta(x, x_0) \leqq r,$$

also to speak of a sphere of radius $e$ as a *vicinity* of its center. The notion $x$ *interior to the set* $E$ can be defined in the following equivalent ways:

(a) there exists a sphere having $x$ as center all of whose points belong to $E$, or

(b) $x$ belongs to $E$ and is not a limiting element of any set consisting of points all of which do not belong to $E$. From the properties of $\delta$ it follows that all points $x$ such that $\delta(x, x_0) < r$ are interior to the sphere center $x_0$ and radius $r$.

In a general metric space the Weierstrass-Bolzano Theorem is not a consequence of the boundedness of a set. Instead we have the property *compact*, a set $E$ being compact, if every infinite subset has a limiting element. If every infinite subset of $E$ has a limiting element in $E$, we shall call $E$ *self-compact*,* which concept in a metric space is equivalent to compact and closed.

As in the linear case, we shall say that a family $\mathfrak{F}$ of sets $I$ chosen from a space $\mathfrak{D}$ covers the set $E$ if every point of $E$ is interior to some set $I$ of $\mathfrak{F}$. Then the Borel Theorem can be stated:

*Any self-compact set $E$ which is covered by a family $\mathfrak{F}$, can be covered by a finite subfamily of $\mathfrak{F}$.*

It may occasionally be useful to refer to the property expressed in this theorem as the *Borel property*, i.e., $E$ has the Borel property if from any family covering $E$, a finite subfamily covering $E$ can be selected.

Fréchet proved the theorem first for the case in which the family is denumerable. The process is very much the same as that given in §2(d), for the linear case. The family is arranged in sequential order $I_1, \cdots, I_n, \cdots$, and by a step-by-step process replaced by an array $I_1, \cdots, I_{nm}, \cdots$,

* Due to E. W. Chittenden, this BULLETIN, vol. 21 (1915), p. 18.

in which each set contains at least one element of $E$ not interior to the preceding set. If this sequence is infinite, we get a sequence of distinct elements of $E$, $x_1, \cdots, x_m, \cdots$, where $x_m$ is interior to $I_{n_m}$ but not to any preceding set. By the self-compactness of $E$, this sequence has a limiting element $x_0$ of $E$. Now $x_0$ is interior to some set $I_n$ and consequently $I_n$ contains a sphere having $x_0$ as center, and hence an infinite number of the sequence $x_m$ as interior points. This leads to a contradiction with the method of selecting $I_{n_m}$ and $x_m$.

We call attention to the fact that the proof utilizes in particular two ideas, (a) the self-compactness of $E$, and (b) the fact that if $x$ is interior to $E$, and a limiting element of $E_1$ then $E$ contains as interior elements an infinite number of elements of $E_1$.

For the general Borel Theorem, Fréchet* originally postulated further properties of the space, viz., that the space $\mathfrak{D}$ is *separable*, i.e., can be considered as the derived set of a denumerable set of its elements. It was shown later that any compact set $E$ of a space $\mathfrak{D}$ has the same property, i.e., for any compact set $E$, there exists a denumerable subset $E_0$ such that $E$ is contained in $E_0+E_0'$, thus removing the restriction of separability for the space $\mathfrak{D}$.†

The proof of the Borel Theorem depends upon the following lemma.

LEMMA.‡ *If every point $x$ of a compact set $E$ is the center of a sphere of radius $r(x)$, and if there exists an $e$ such that for all $x$ of $E$*

$$r(x) > e > 0,$$

*then all points of $E$ are interior to a finite number of these spheres.*

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* Loc cit., pp. 25–27.

† See T. H. Hildebrandt, AMERICAN JOURNAL, vol. 34 (1912), pp. 278–281; W. Gross, WIENER BERICHTE, vol. 123 IIa(1914), pp. 809–812; Fréchet, BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 45 (1917), pp. 1–8.

‡ See Hahn, *Reelle Funktionen*, 1921, pp. 89–93. See also Urysohn, FUNDAMENTA MATHEMATICAE, vol. 7 (1925), pp. 46–48, where the role of the axiom of choice in the proof of the Borel Theorem is emphasized.

Let $S(x_1)$ be the sphere about any point $x_1$ of $E$, and $x_n$ any point not interior to $S(x_1)$, $S(x_2)$, $\cdots$, $S(x_{n-1})$. Then on account of the fact that $\delta(x_n, x_m) > e$ for every $n$ and $m$ and $E$ is compact, the sequence $x_n$ is finite.

With this lemma, the Borel Theorem in a metric space can be proved in two ways. Either (a) following the method of §2(c) on the linear interval, let $r(x)$ be the least upper bound of the radii of the spheres of center $x$ interior to some set $I$ of the given family. The self-compactness of the family yields immediately that the $r(x)$ have a positive lower bound, and the lemma then suggests a method for selecting the finite subfamily from $\mathfrak{F}$.

Or (b) from the lemma we conclude that for every $n$ the points of a compact set $E$ are interior to a finite number of spheres of radii $1/n$. For every $n$ then we retain the spheres which are interior to some set $I$ of $\mathfrak{F}$. We obtain in that way a denumerable family of spheres, covering $E$, and hence by the denumerable-to-finite Borel Theorem, we can select a finite subfamily of spheres, which in turn defines a finite subfamily of $\mathfrak{F}$ covering $E$.

This last proof contains practically the proof of the result that *any compact set $E$ is separable.* For the centers of the spheres of radius $1/n$ having $E$ as interior points will be a denumerable set $E_0$ having the property that $E$ belongs to $E_0 + E_0'$.

Also there is present a special case of the Lindelöf Theorem in a metric space, viz.

*If in a space $\mathfrak{D}$ $E$ is any separable set covered by a family $\mathfrak{F}$ of sets $I$, then it is covered by a denumerable subfamily.*

Obviously the reason why the Lindelöf theorem holds in linear or $n$-dimensional space is because these spaces are separable. The method of proof is entirely similar to these special instances. Let $E_0 = [x_n]$ be a denumerable subset of $E$ such that $E$ belongs to $E_0 + E_0'$. Consider the denumerable family of spheres, center $x_n$, and rational radii interior

to some $I$ of $\mathfrak{F}$. They will cover $E$ and set up a one to one correspondence with a denumerable subfamily of $\mathfrak{F}$.

The condition that $E$ is self-compact is necessary for the validity of the Borel Theorem. For if $E$ is not self-compact, then there exists a denumerable set of elements $[x_n]$ of $E$ without a limiting element in $E$. To every point $x$ of $E$ there corresponds then a sphere containing at most one point of $[x_n]$. These spheres form a family covering $E$, but every finite subfamily contains only a finite number of points of $[x_n]$ and hence does not cover $E$.

Similarly the condition that $E$ be separable is also necessary for the Lindelöf Theorem. Instead of proving this directly we relate this theorem and separability to a third property suggested in Lindelöf's paper. If we define an *element of condensation* of a set $E$, as a limiting point of $E$ which remains a limiting point after the removal of any denumerable set from $E$, and a set as *self-condensed* if every non-denumerable set chosen from $E$ has at least one element of condensation in $E$, then we have the equivalence of the following three properties in a metric space $\mathfrak{D}$:*

(A) *The set $E$ is separable.*

(B) *The set $E$ is self-condensed.*

(C) *The set $E$ has the Lindelöf property*, i. e., if $\mathfrak{F}$ is any family of sets covering $E$, then a denumerable sub-family of $\mathfrak{F}$ covers $E$.

We have already shown that (A) implies (C). To show that (C) implies (B) follows the lines of the converse of the Borel Theorem above; the assumption of a non-denumerable set $E_0$ without an element of condensation in $E$ yields via (C) a denumerable set of spheres each of which contains only a denumerable set of elements of $E_0$. To show finally that (B) implies (A), we show first that if $E$ is condensed and if each point of $E$ is the center of a sphere of radius greater than $e$, then all the points of $E$ are interior to a denumerable

* See Gross, loc. cit., pp. 805–12; Fréchet, ANNALES DE L'ÉCOLE NORMALE, (3), vol. 38 (1921), pp. 349–356.

subset of these spheres. Allowing $e$ to take on successively the values $1/n$, we get a denumerable set of centers of spheres which is the desired denumerable subclass of $E$.

In so far as measure has not yet been effectively connected with a general metric space, it is not possible to give generalizations of the Vitali Theorem. However K. Menger* has given some results which are comparable to the Vitali Theorem.

We define the *diameter* of a set $I$ as the diameter of the minimum sphere containing $I$. Then Menger is interested in the question: What properties of a set $E$ in a metric space and what properties of a family $\mathfrak{F}$ of sets $I$ such that every point of $E$ is interior to a subfamily of sets $I$ whose diameters converge to zero, are sufficient to make possible the selection of a finite or denumerable subfamily of $\mathfrak{F}$ of sets $I$ whose diameters approach zero, and which covers $E$.

If $d(I)$ is the diameter of $I$ and $d(I, x)$ is the least upper bound of the diameters of spheres center $x$ contained in $I(x)$, then Menger's principal result is that the selection desired is possible provided

(a) $E$ is the sum of a denumerable set of compact sets;

(b) if $I(x)$ are the sets of $\mathfrak{F}$ associated with $x$, then for every $x$ the condition that $d(I)$ approach zero is a consequence of the fact that $d(I, x)$ approaches zero. This latter condition is fulfilled in case there exists a positive-valued function $f(x)$ on $E$ such that for each $I$ associated with $x$

$$d(I) < f(x) \cdot d(I, x) .$$

The proof is made first for a compact set $E$. If $E_n$ is the subset of $E$ for which there exists an $I$ such that $d(I, x) > 1/n$ then $E_n$ is interior to a finite number of these sets $I$. As a consequence $E$ can be covered by a finite or denumerable family of sets $I_n$ from $\mathfrak{F}$, which if denumerable has the property that there exists a point $x_n$ of $I_n$ such that $d(I_n, x_n)$ approaches zero, so that $d(I_n)$ converges to zero

* *Einige Überdeckungssaetze der Punktmengenlehre*, WIENER BERICHTE, vol. 133IIa (1924), pp. 421–444.

as desired. The extension to the denumerable set of compact sets is obvious.

In addition to having contact with the Vitali Theorem, the results of Menger are also related to Young's Tile Theorem (§6) and R. L. Moore's reduction theorem (§5). The conditions of Menger are sufficient. No doubt further interesting results can be obtained by considering necessary conditions.

9. *The Borel Theorem in a Space $\mathfrak{L}$ with Limit of Sequence defined.* Fréchet's thesis besides considering metric spaces also postulated a more fundamental space $\mathfrak{L}$, that in which limit of a sequence is defined. Limit is subject to three conditions: (1) the limit of a sequence is unique, (2) the limit of a sequence consisting of the same element repeated is this element, and (3) any subsequence of a sequence having a limit has the same limit. Obviously limiting element, interiority, and the other concepts can be defined in the main as suggested for metric spaces.

The first statement of a Borel Theorem in a space $\mathfrak{L}$ is due to E. R. Hedrick.* By analyzing the proof for the Borel Theorem in the denumerable-to-finite case, he observed that it could be effected provided the space $\mathfrak{L}$ had the following property† (called by Fréchet the Hedrick property):

(H) *If $x$ is interior to a set $E$, then an infinite number of elements of any sequence having $x$ as limit are interior to $E$.*

He found that this property was a consequence of the simpler property:

(S) *The derived class of any class is closed.*

For suppose $x$ is interior to $E$ and the limit of a sequence of distinct elements $x_n$, the result being obvious if the elements are not distinct. We show that there exists an $n_0$

* Transactions of this Society, vol. 12 (1911), pp. 285–7.

† Fréchet's statement (cf. Annales de l'École Normal, (3), vol. 38 (1921), p. 348) of this property is: If $x$ is interior to $E$ and a limiting element of $F$, then $x$ is a limiting point of a subset $F_0$ of $F$ consisting entirely of elements interior to $E$. In an $\mathfrak{L}$ space this statement is equivalent to the one above.

such that for $n>n_0$, $x_n$ is interior to $E$. The assumption of the contrary gives rise to a sub-sequence $\{x_{n_m}\}$ no point of which is interior to $E$, i. e., each member of the sub-sequence is the limiting element of a sequence of elements $\{x_{n_m k}\}$ not members of $E$. If we let $E_0$ be the class of elements $[x_{n_m k}]$, then $E_0'$ contains the sequence $\{x_{n_m}\}$ and consequently the point $x$, by the property S. The element $x$ would then be a limiting element of $E_0$ which would contradict the interiority condition of $x$ to $E$.

It is now fairly obvious that the Borel Theorem is valid in the form

*If the space $\mathfrak{L}$ has the property* S *then any self-compact set $E$ covered by a denumerable family $\mathfrak{F}$ of sets $I$, is covered by a finite sub-family of $\mathfrak{F}$*, i. e., any self-compact set $E$ has the denumerable-to-finite Borel property.

The condition that $E$ be self-compact is necessary in any space $\mathfrak{L}$.* For suppose $\{x_n\}$ is a sequence chosen from $E$ not having a limiting element in $E$. Let $I_m$ be the set obtained from the given space by deleting the elements $x_n$ for $n>m$. Then obviously the set $E$ is covered by the family $\mathfrak{F}$ of $I_m$, since no element of $E$ is a limiting element of $\{x_n\}$, but no finite subfamily of $\mathfrak{F}$ contains all points of the sequence $\{x_n\}$.

Further the property S and so H is a necessary property in a space $\mathfrak{L}$ for the Borel Theorem in this form: if $E$ is self-compact then $E$ has the denumerable-to-finite Borel property.† Let if possible the set $E$ be such that $E'$ is not closed. Then there exists a sequence $\{x_n'\}$ of elements of $E'$ with a limit $x''$ not belonging to $E'$. Let the elements $\{x_{mn}\}$ of $E$ be such that $x_n'$ is a limiting element of $x_{mn}$ for $m=1, 2, \cdots$. Consider the family $\mathfrak{F}$ consisting of (a) $I_0$, the set remaining after removing the elements $x_{mn}$ from the fundamental space, and (b) $I_n$ the set remaining after

* Cf. E. W. Chittenden, *The converse of the Heine-Borel theorem in a Riesz domain*, this BULLETIN, vol. 21 (1915), pp. 179–183.

† Cf. Fréchet, BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 45 (1917), pp. 1–8; Chittenden, this BULLETIN, vol. 25 (1918), pp. 60–66.

removing all members of the sequence $\{x_n'\}$ excepting $x_n'$ from the fundamental space. Then the self-compact set $E_0$ consisting of the sequence $\{x_n'\}$ and $x''$ is covered by the family $\mathfrak{F}$ but by no finite subfamily of this family.

We note that *in a space* $\mathfrak{L}$ *there is then equivalence between these three properties*:

(S) *The derived class of every class E is closed.*

(H) *If x is interior to E, and is the limiting element of a set* $E_1$ *then an infinite subset of* $E_1$ *is interior to E.*

(B) *If E is self-compact, it has the denumerable-to-finite Borel property.*

The problem of determining conditions under which the any-to-finite Borel Theorem is valid in a space $\mathfrak{L}$ remained unsolved for some time. Obviously the methods of metric space could not be used. A method of attack is suggested by the proof in the case just treated, viz. to utilize the theory of transfinite ordinals. The first solution of the problem was given by R. L. Moore.* He calls a *monotonic family of classes G*, a family such that for each pair $G_1$ and $G_2$ of the family, one contains the other. He then defines the concept called by Fréchet *perfectly compact.* The set $E$ is perfectly compact in case every infinite monotonic family of sets chosen from $E$ or the family of their derived sets has a common element. That a perfectly compact set is compact is obvious from a consideration of the monotonic family where $G_m$ consists of the elements of the sequence $\{x_n\}$ for $n>m$.† The relationship of the property "perfectly compact" to the theorem "any monotonic sequence of closed compact sets has a common element" will appear later. Moore's result is as follows.

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* See PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, vol. 5 (1919), pp. 206–210.

† Fréchet (ANNALES DE L'ÉCOLE NORMALE, (3), vol. 38, pp. 334–6) shows that in a metric space every compact set is also perfectly compact. This can also be deduced from the converse of the any-to-finite Borel Theorem in a metric space and in an $\mathfrak{L}$ space.

*In a space $\mathfrak{L}$ with the property* **S**, *a necessary and sufficient condition that $E$ have the any-to-finite Borel property is that $E$ be perfectly self-compact.*

Suppose there is given a family $\mathfrak{F}$ of sets $I$ which covers $E$. Well-order this family, and then by a step-by-step process reduce the family so that each $I$ contains as interior point at least one element of $E$ not interior to any of the preceding sets in the array. Let the members of the resulting family be $I_1, I_2, \cdots, I_\alpha, \cdots$, and let $x_\alpha$ be interior to $I_\alpha$ but not to $I_\beta$ for $\beta > \alpha$. Let $E_\alpha$ be the set of points $x_\beta$ for $\beta > \alpha$. Then the family consisting of the $E_\alpha$ is a monotonic family from $E$, and since the $E_\alpha$ do not have a common point, either the family is finite or the derivatives $E_\alpha$ have a common point $x'$. In the latter case there exists a $\gamma$ such that $x'$ is interior to $I_\gamma$, and since $x'$ is a limiting element of $E_\gamma$, by the property H it follows that $I_\gamma$ contains interior points of the set of $x_\alpha$ for ordinals greater than $\gamma$, contrary to the definition of the $x_\alpha$. Hence the number of members of the family of $E_\alpha$ is finite.

For the proof of the converse, let $E$ be a set of the space $\mathfrak{L}$ having the any-to-finite Borel property. Then as shown above $E$ is self-compact, i. e. compact and closed. Let $G$ be any infinite monotonic family of sets $E_\alpha$ drawn from $E$. Then since $E$ is closed $E_\alpha'$ will belong to $E$. Moreover the property S insures that the sets $\overline{E}_\alpha = E_\alpha + E_\alpha'$ are closed. Let the sets $I_\alpha$ of the family $\mathfrak{F}$ consist of the points remaining after deleting the elements of $\overline{E}_\alpha$ from the fundamental set. If the sets $\overline{E}_\alpha$ have no common element, it follows that $E$ will be covered by the family $\mathfrak{F}$, but a finite subfamily of $\mathfrak{F}$ will not cover $E$ since the sets $\overline{E}_\alpha$ contain elements for each $\alpha$.

Another solution of this question of the Borel any-to-finite Theorem in a space $\mathfrak{L}$ was given in 1923 by Kuratowski and Sierpinski.* In so far, however, as their result is practically stated in a space in which vicinities are defined we shall postpone consideration of it to a later section of this paper.

* *Le théorème de Borel-Lebesgue dans la théorie des ensembles abstraits*, FUNDAMENTA MATHEMATICAE, vol. 2 (1921), pp. 172–8.

10. *Vicinity Spaces* $\mathfrak{V}$. *Hausdorff Form of the Borel Theorem.* The first suggestions of sets or vicinities as the basis for consideration in a general space are to be found in the paper of Hedrick.* Another development based to some extent on the Riesz† postulates for limiting element was given by R. E. Root.‡ About the same time, Hausdorff in his book on Mengenlehre developed systematically the point set theory in a vicinity space. Later, in 1918, Fréchet§ gave an independent development of the same type of space showing in particular the relationship between the space characterized by the Riesz postulates and a space based upon vicinities. The Hausdorff postulates have come to be accepted as a satisfactory basis, and a space based on them is usually called a topologic space. The postulates are as follows:

I. To every point $x$ there corresponds a family of sets $V(x)$, chosen from the given space, and containing $x$.

II. If $V_1(x)$ and $V_2(x)$ are vicinities of $x$, then there exists a common subvicinity $V_3(x)$.

III. For every pair of points $x_1$ and $x_2$, there exist vicinities $V_1(x_1)$ and $V_2(x_2)$ without common elements.

IV. If $x_2$ belongs to $V_1(x_1)$ then there exists a $V_2(x_2)$ contained in $V_1(x_1)$.

In a vicinity space *limiting element* of a class $E$ can be defined either (a) $x$ is a limiting element of $E$ if every $V(x)$ contains at least one point other than $x$, or (b) $x$ is a limiting element of $E$ if every $V(x)$ contains an infinity of

* Loc. cit., p. 289; Fréchet, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), pp. 320–4, showed that the space postulated by Hedrick was a metric space.

† See ATTI DEL IV CONGRESSO INTERNAZIONALE (Roma) 1909, vol. 2, pp. 18–22.

‡ Cf. TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), pp. 51–70.

§ BULLETIN DES SCIENCES MATHÉMATIQUES, (2), vol. 42 (1918), pp. 138–156; called Fréchet I in the sequel. Fréchet considers a type of space that he has called "espace accessible," which is equivalent to a vicinity space subject to postulates similar to those of Hausdorff, IV and especially III being replaced by weaker ones.

elements. In a $\mathfrak{B}$ space satisfying conditions I, II, and III these two definitions are equivalent.

More generally we can define *$x$ is a limiting element of $E$ of power $\mu$*, if every vicinity of $x$ contains a subset of power $\mu$. Finally $x$ is called a *complete limiting element of $E$* if every $V(x)$ contains a subset of $E$ of the same power as $E$.

It is obvious that we can obtain an $\mathfrak{L}$ space in a $\mathfrak{B}$ space by assuming that $\lim_n x_n = x$ is defined "for every $V(x)$ there exists an $n_0$ such that if $n > n_0$ then $x_n$ is contained in $V(x)$." But limiting element based on the sequence notion of this $\mathfrak{L}$ space need not agree with the limiting element of the given $\mathfrak{B}$ space, unless the $\mathfrak{B}$ space is subjected to additional conditions.* On the other hand given an $\mathfrak{L}$ space it is possible to define it as a $\mathfrak{B}$ space in which limiting elements are the same.†

The notion *interiority* can be defined in different ways, equivalent if we are in a Hausdorff $\mathfrak{B}$ space (i. e. subject to conditions I, II, III and IV)‡: *$x$ is interior to $E$* if $x$ belongs to $E$ and either (a) every set $E_1$ having $x$ as a limiting element, contains at least one element other than $x$ of $E$, or (b) every set $E_1$ having $x$ as limiting element contains an infinity of elements of $E$, or (c) there exists a $V(x)$ containing only elements of $E$.

Obviously $x$ is interior to every $V(x)$.

An *open set* or *region* is a set containing only interior points. On account of condition IV every $V(x)$ is an open set. We note that the sum of two open sets is open, also the complementary set of a closed set is open.

Finally we note that on account of condition IV, a Hausdorff $\mathfrak{B}$ space has the property S.

The outstanding difference in Hausdorff's statement of the Borel Theorem from that stated by Fréchet is that the

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* Cf. Root, loc. cit., pp. 67–71; Fréchet, I, p. 148.

† Cf. Fréchet, I, pp. 140–148.

‡ Limiting element may be of power 2 or $\aleph_0$. Definition (c) is perhaps most satisfactory in so far as limiting element does not enter directly.

family $\mathfrak{F}$ of covering sets consists of open sets, which introduces an element of simplicity, since belonging to an open set is equivalent to being interior. The most elegant form of the Borel Theorem in a Hausdorff $\mathfrak{B}$ space has been given by P. Alexandroff and P. Urysohn,* by calling attention to the following equivalences:

THEOREM I. *The following properties of a set E in a Hausdorff space are equivalent:*

$A_0$. *E is self-compact.*

$A_1$. *Every denumerable subset of E has a complete limiting element in E.*

B. *If $\mathfrak{F}$ is a denumerably infinite monotonic family of closed sets $F_n$ such that each $F_nE$ contains at least one element, then the sets $F_nE$ have a common element.*

C. *If E is covered by a denumerable family $\mathfrak{G}$ of open sets G, then E is covered by a finite subfamily of $\mathfrak{G}$.*

It is obvious that $A_0$ and $A_1$ are equivalent. The equivalence of B and C is a matter of taking complementaries, products and sums. Assume B, and let $\mathfrak{G}=[G_n]$. Then the sets $\sum_1^m G_n$ are open sets, and the sets $\mathfrak{B}-\sum_1^m G_n$ closed, as a matter of fact form a monotonic family. If the sets $E(\mathfrak{B}-\sum_1^m G_n)$ contain an element for every $m$, then by B they have a common element, i. e. the family $\mathfrak{G}$ does not cover $E$. Conversely, assume C and a monotonic family of closed sets $[F_n]$ such that each $F_nE$ contains a point. Then the $\mathfrak{B}-F_n$ are open sets. If the $F_nE$ do not have a common point, then the sets $\mathfrak{B}-F_n$ will contain all points of $E$. The finiteness of the equivalent set chosen from the $\mathfrak{B}-F_n$ leads to a contradiction.

The fact that $A_1$ implies B is a well known result. The denumerable family $F_nE$ leads to a set $E_0$, of points $x_n$ where $x_n$ belongs to $F_nE$. The complete limiting element of $E_0$ in $E$ is also a limiting element for each $F_nE$, and hence in each member of the family.

* Cf. MATHEMATISCHE ANNALEN, vol. 92 (1924), pp. 258–60.

To complete the equivalence we show that C implies $A_0$. If possible let $E_0$ be a denumerable set without a limiting element in $E$. Then $\bar{E}_0 = E_0 + E_0'$ will be closed and have no limiting elements in $E$. Then if $x_n$ is any element of $E_0$, the sets

$$G_n = \mathfrak{D} - \bar{E}_0 + x_n$$

will be open and will contain all points of $E$, but a finite subfamily of the $G_n$ will not.

More generally we have the following theorem:

THEOREM II. *The following properties of $E$ in a Hausdorff $\mathfrak{B}$ space are equivalent:*

A. *Every infinite subset of $E$ has a complete limiting element belonging to $E$.*

B. *If $\mathfrak{F}$ is any well-ordered monotonic decreasing family of closed sets $\mathfrak{F}$ such that for each $\alpha$, $F_\alpha E$ contains an element, then the sets $F_\alpha E$ have a common element.*

C. *If $E$ is contained in a family $\mathfrak{G}$ of open sets $G$, then it is contained in a finite subfamily.*

The proof of the equivalence of B and C follows the lines of the corresponding proof for Theorem I.

Assume A, and let $\mathfrak{G}$ be a family of open sets $G$.* Let $\mu$ be the power such that any subfamily of $\mathfrak{G}$ of power less than $\mu$ does not contain $E$, but there are subfamilies of power $\mu$ containing $E$. Let $\mathfrak{G}_0$ be a subfamily of $\mathfrak{G}$ of power $\mu$ containing $E$. Let $\Omega$ be the least ordinal of power $\mu$. Then we can well-order $\mathfrak{G}_0$ in the form $G_1, G_2, \cdots, G_\alpha, \cdots$ such that $\alpha < \Omega$ and by possible deletion assume that each $G_\alpha$ contains at least one point $x_\alpha$ of $E$ not in any preceding set. Consider the set $E_0 = [x_\alpha]$. By the property A if $E_0$ is infinite, $E_0$ has a complete limiting element $x$, belonging to $E$, and consequently to some member $G_\beta$ of the family $\mathfrak{G}_0$. Consequently every vicinity of $x$ and so also $G_\beta$ will contain a subset of $E_0$ of power $\mu$. Since the set of elements $x_\alpha$ for $\alpha < \beta$ is of power less than $\mu$ it follows that $G_\beta$ contains

* This method of proof follows the lines suggested by Kuratowski and Sierpinski, loc. cit., pp. 174–5.

points of $E_0$ of index greater than $\beta$, contrary to the method of choice of the $x_\alpha$. Hence the set $E_0$ is finite.

To complete the equivalence we show that C implies A. For suppose $E_0$ is an infinite subset of $E$ not having a complete limiting point in $E$. Then for every $x$ of $E$ there exists a vicinity $V(x)$ such that the power of $V(x)E_0$ is less than that of $E_0$. These vicinities being open sets, constitute a family of open sets containing $E$. But obviously a finite subfamily cannot contain all points of $E_0$.

Theorem I suggests an extension in which the word "denumerable" is replaced by "power less than or equal to $\mu$," in the properties $A_1$, B and C, the proof being similar. Theorem II suggests the following extension.

THEOREM III. *The following properties of a class $E$ in a Hausdorff $\mathfrak{B}$ space are equivalent:*

A. *Every infinite set of power $\mu$ has a complete limiting element in $E$.*

B. *If $\mathfrak{F}$ is a well-ordered monotonic decreasing family of closed sets $F$, of power $\geqq\mu$, such that for each $\alpha$, $F_\alpha E$ contains at least one element, then the sets $F_\alpha E$ have a common element.*

C. *If $E$ is contained in a family $\mathfrak{G}$ of open sets $G$, the power of $\mathfrak{G}$ being $\geqq\mu$, then $E$ is covered by a subfamily of $\mathfrak{G}$ of power $\leqq\mu$.*

This theorem contains among others the Lindelöf Theorem as a special case. The proof is similar to that of Theorem II. It is interesting that in the general space the Lindelöf Theorem and the Borel Theorem seem to join hands, a fact not to be foreseen by a consideration of $n$-dimensional space.

Alexandroff and Urysohn call a set $E$ satisfying the conditions of Theorem II *bicompact*, because it is a meeting place of the generalization of Theorem I and of Theorem III. It is obvious however that property B is a special case of the *perfectly compact* property. Perhaps *completely compact* would be a better term.

It remains to consider to what extent the Hausdorff postulates on the $\mathfrak{B}$ space are needed. An analysis of the proofs shows that in any space in which closed and open sets are complementary the properties B and C are equivalent.*

In Theorem I, $A_0$ and $A_1$ are equivalent, and imply B and C if the $\mathfrak{B}$ space is subject only to conditions I, provided "limiting element" is the limiting element of power $\aleph_0$ (i. e. definition (b)). The converse that $A_0$ and $A_1$ follow from B and C requires condition IV, making $V(x)$ an open set, which condition is practically the property S: the derived set of a set is closed. If "limiting element" is of power 2, (i. e. definition (a)), then B and C follow from $A_1$, but $A_0$ and not $A_1$ follows from B and C under additional postulate IV.

Theorems II and III are true under a space satisfying postulates I and IV, the latter being required in the proof of the result "C implies A," for instance.

11. *General Borel Theorem in a $\mathfrak{B}$ Space.* We have pointed out that the Hausdorff statement of the Borel Theorem is based on families of open sets. It seems desirable to consider briefly what happens in case we are dealing with families of arbitrary sets, the deciding covering property being then interiority. At the same time, the attempt is to reduce the properties of the $\mathfrak{B}$ space to a minimum.

For most of this section, we shall assume considerations based on a $\mathfrak{B}$ space subject to condition I of Hausdorff. *Limiting element* is of power 2, i. e., $x$ is a limiting element of $E$ in case every vicinity of $x$ contains a point of $E$ other than $x$. Further *$x$ is interior to $E$* if $x$ belongs to $E$ and every set having $x$ as limiting element has a point other than $x$ in common with $E$, or if $E$ contains a vicinity of $x$. Compactness, derived sets, limiting points of power $\mu$, complete limiting points are defined as above. We use finally a new concept: $x$ is a *complete interior limiting point*†

* Cf. Saks, FUNDAMENTA MATHEMATICAE, vol. 2 (1921), pp. 1–3.

† Chittenden (this BULLETIN, vol. 30 (1924), p. 556, referred to as Chittenden I in the sequel), calls such a point a hypernuclear point.

of $E$ if every vicinity of $x$ contains as *interior* points a set of the same power as $E$. The definition of interior limiting point of power $\mu$ is then obvious.

As in the previous section it seems proper to consider the following properties of a set $E$ in this space as being connected with the denumerable-to-finite Borel Theorem:

$A_0$. $E$ is self-compact.

$A_1$. Every denumerable subset of $E$ has a complete limiting point in $E$.

$A_2$. Every denumerable subset of $E$ has a complete interior limiting point in $E$.

B. If $\mathfrak{F}$ is a monotonic denumerable family of sets $F$ chosen from $E$, either the sets $F$ or their derived sets have a common element in $E$.

C. If $E$ is covered by a denumerable family $\mathfrak{G}$ of sets $G$, then it is covered by a finite sub-family of $\mathfrak{G}$.

We obviously have that $A_2$ implies $A_1$ implies $A_0$.

The statement "$A_2$ implies C" is a form of the Borel Theorem whose proof is obvious. The converse C implies $A_0$ can be proved by assuming if possible $E$ not self-compact. Then there exists a denumerable set $E_0$ of elements of $E$ without a limiting element in $E$. Then for each point $y_n$ of $E_0$ there exists a $V(y_n)$ containing only the point $y_n$ of $E$, and for each point $x$ of $E$ not a point of $E_0$ a vicinity $V(x)$ containing no points of $E_0$. Then the sets

$$G_0 = \sum V(x),\ G_n = V_n(y_n)$$

cover $E$ but a finite subfamily does not.

The proof of the fact that $B$ follows from $A_1$ can be modelled after the more general result given below. The converse is is obvious.

To obtain further results it seems necessary to add additional hypotheses on the fundamental space.

The assumption that $\mathfrak{B}$ satisfies the condition IV of Hausdorff, or has the property $S$, that derived classes are closed, gives a Hedrick property which can be stated as follows.

(H) *If $x$ is interior to $E$ and a limiting element of $E_0$ of power $\mu$ then $E_0E$ contains a set of interior elements, whose power is $\mu$.*

If derived classes are closed, then according to Fréchet* every vicinity $V(x)$ of $x$ contains a subvicinity $V_0(x)$ all of whose elements are interior to $V(x)$. Now if $x$ is interior to $E$, then there exists a vicinity $V(x)$ of $x$, which contains only points of $E$. The subvicinity $V_0(x)$ of $V$ then defines a subset of $E_0$ of power $\mu$ all points of which are interior to $E$.

If $\mathfrak{B}$ has the property S and consequently H, then we can state the equivalence of B and C. The proof of this equivalence follows the lines of proof of Moore's form of the Borel Theorem in an $\mathfrak{L}$ space.†

The other possible equivalences seem to be linked up with the fact that in the general $\mathfrak{B}$ space with limiting element as defined, a finite class may have a limiting element. The conditions II and III of Hausdorff are sufficient to guarantee the contrary. Under these conditions it is possible to prove that C implies $A_2$ and $A_0$ implies $A_1$. The proof of the former of these statements follows the lines of the proof of the fact that C implies $A_0$ given above. The latter is obvious. We have as a consequence the following theorem.

THEOREM I. *If the $\mathfrak{B}$ space satisfies condition* IV *of Hausdorff, then properties* $A_1$, B *and* C *are equivalent. If the space is a Hausdorff $\mathfrak{B}$ space, then all the properties* $A_0$, $A_1$, $A_2$, B, *and* C *are equivalent.*

The extension of these results to the case where the word "denumerable" is replaced by "power less than or equal to $\mu$" in the properties $A_1$, $A_2$, B, and C is obvious.

Theorem II of § 10 suggests the consideration of the following properties:

* Cf. Fréchet, I, p. 145.

† Chittenden (I, p. 519) has shown that in a general $\mathfrak{B}$ space the property S is not a consequence of the Borel Theorem in the form "$A_0$ implies C."

$A_1'$. Every subset of $E$ has a complete limiting element in $E$.

$A_2'$. Every subset of $E$ has a complete interior limiting element in $E$.

B'. $E$ is perfectly self-compact, i. e., if $\mathfrak{F}$ is any monotonic family of sets $F$ chosen from $E$ then either the sets $F$ or their derived sets have a common element in $E$.

$C'$. $E$ has the Borel any-to-finite property, i. e., if $E$ is covered by a family $\mathfrak{G}$ of sets $G$ then $E$ is covered by a finite subfamily of $\mathfrak{G}$.

Chittenden* has stated the following theorem.

THEOREM II. $A_1$ *is equivalent to* B' *and* $A_2'$ *is equivalent to* C'.

We prove first that $A_1'$ implies B'. Suppose $\mathfrak{F}$ a monotonic family of sets $F$ chosen from $E$. Let $H$ be a set of elements of $E$ such that every element of $H$ is in some $F$ and every $F$ contains an element of $H$. Well-order $H$, $x_\alpha$ corresponding to the ordinal $\alpha$. Then we determine a well-ordered subfamily of $\mathfrak{F}$ and a subset $H_0$ of $H$ by the requirement that $F_\alpha$ contain no element of $H$ with ordinal $\gamma < \beta_\alpha$ and $x_{\beta_\alpha}$ be the first element of $H$ common to all $F_\gamma$ for $\gamma \leqq \alpha$. Then for every $F$ of $\mathfrak{F}$ there exists an $\alpha$ such that $F$ contains $F_\alpha$. For $F$ contains an element $x_\gamma$ of $H$, and for some $\alpha$, $x_\gamma$ will not belong to $F_\alpha$, so that by the monotonic character of $\mathfrak{F}$, $F$ contains $F_\alpha$. From this it follows that if the derived sets $F_\alpha'$ have a common element, the same will be true of the sets $F'$, i.e. if the theorem holds for a well-ordered monotonic family, it holds for any monotonic family.

* I, pp. 514–8. Very recently, since this paper was in type, Professor Chittenden has called my attention to the fact that the proofs of the first part of this theorem contained in his paper are not correct. Correct proofs by Sierpinski, who first noted the error, will appear in the next issue of this BULLETIN. The proof that is given here of the fact that $A_1'$ implies $B'$ is a modification of Chittenden's proof, obtained before I was aware of an error in his work.

Let $F_1, \cdots, F_\alpha, \cdots$ be the elements of $\mathfrak{F}$ in order. Since every well-ordered set without final element is cofinal with a regular ordinal number,* it is obviously sufficient to prove the theorem for the case where every ordinal $\alpha$ precedes $\Omega$ the least transfinite ordinal corresponding to $\mu$ the power of $\mathfrak{F}$. Let $x_\alpha$ belong to $F_\alpha$ but not to $F_{\alpha+1}$. Then from the assumption concerning $E$ it follows that the set $H$ of elements $x_\alpha$ will have a complete limiting element $x'$. Every vicinity of $x'$ contains a set of $H$ of power $\mu$, i.e. for every $\alpha$ an element $x_\beta$ with $\beta>\alpha$, and consequently an element of $F_\alpha$. Hence $x'$ is common to the sets $F_\alpha'$.

For the proof of the converse, that $B'$ implies $A_1'$, we refer the reader to the paper by Sierpinski, which will appear in the next issue of this BULLETIN.

The proof of the equivalence of $A_2'$ and $C'$ is parallel to the corresponding equivalence in Theorem II of § 10.

Apparently to get complete equivalence it is necessary either to strengthen condition B′ or add further postulates on the fundamental space. Just what is necessary has not as yet been determined. A sufficient, but not necessary condition is that the space have the property S. Then the property H is valid, which added to B′ gives C′ as in Moore's proof of the Borel Theorem in an $\mathfrak{L}$ space given in § 9. We thus have the following theorem.

THEOREM II′. *If the space $\mathfrak{B}$ has the property* S, *then properties* $A_1'$, $A_2'$, B′, *and* C′ *as applied to any set E of the space are equivalent.*

We note that since in a space $\mathfrak{B}$ with the property S, the set $\bar{E}=E+E'$ is closed, perfect self-compactness or condition B′ is equivalent to "If $\mathfrak{F}$ is any infinite monotonic family of closed sets such that each set $F\cdot E$ contains an element, then the $FE$ have a common element", the relationship of which to condition B of Theorem II of § 10 is obvious.

Kuratowski and Sierpinski in a space $\mathfrak{L}$ define a $\mathfrak{B}$ space by the condition that any set $V$ to which $x$ is interior is

* Cf. Hausdorff, *Mengenlehre*, 1914, p. 132.

a vicinity of $x$. Then their statement of the Borel Theorem is as follows:

*In an $\mathfrak{L}$ space having the property* S, *a necessary and sufficient condition that every self-compact set $E$ have the Borel any-to-finite property is that every set of the space which is compact and whose derived set is compact have a complete interior limiting element.*

The relationship to Theorem II is apparent.

Finally it is obvious that Theorem II can be generalized as in § 10, giving a theorem corresponding to Theorem III, which contains the Lindelöf Theorem* as a special case, and might be labelled "The Borel-any-to-less-than-power $\mu$ Theorem."

In closing we cannot refrain from calling attention to a justification of the consideration of general spaces, in gathering under the same roof such apparently diverse results as the Borel and Lindelöf Theorems, and producing a result of greater scope.

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* For an $\mathfrak{L}$ space, first given by Kuratowski and Sierpinski, loc. cit., pp. 176–8.

# SOME RECENT WORK IN THE CALCULUS OF VARIATIONS*

BY ARNOLD DRESDEN

## I. The Simplest Problem.

1. *First Order Conditions.* The simplest problem of the calculus of variations is concerned with an integral of the form

$$\int_{x_1}^{x_2} f(x,y,y')dx, \tag{1}$$

in which $f$ is a function of class $C'''$† in a region $R_1$ of 3-space, determined by the conditions that $(x, y)$ be in some region $R$ of the $xy$-plane and $y'$ be finite. The problem may then be formulated as follows.

To determine among all functions $y=y(x)$, $x_1 \leqq x \leqq x_2$, which (1) are of class $C'$; (2) satisfy the conditions $y(x_i)=y_i$, $i=1, 2$; and (3) are such that the points $(x, y(x))$ for $x_1 \leqq x \leqq x_2$ lie in the region $R$, a function $y_0(x)$ for which there exists a positive number $d$, such that all functions $Y(x)$, $x_1 \leqq x \leqq x_2$, which satisfy conditions (1), (2), and (3), and the further condition that $|Y(x)-y_0(x)| \leqq d$ for $x_1 \leqq x \leqq x_2$, give to the integral (1) a value not less (not greater) than the value which this integral has for $y_0(x)$. Here it is understood that $y'$ denotes the derivative $dy/dx$. A function which satisfies the conditions (1), (2), and (3) is called an *admissible function.*

The classical procedure in solving this problem is as follows. We suppose that we have found a solution $y=y_0(x)$,

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* An address presented at the request of the program committee at the Western Meeting of the Society, Chicago, April 2, 1926.

† A function is said to be of class $C^{(n)}$ if it possesses continuous derivatives of orders $1, 2, \cdots, n$; a function is said to be of class $D^{(n)}$, if it is continuous, and consists of a finite number of parts of class $C^{(n)}$.

$x_1 \leqq x \leqq x_2$, of the problem. We then set up a *family of admissible functions*

$$y = y_0(x) + \epsilon\eta(x), \tag{2}$$

in which $\eta(x)$ and $\epsilon$ are so restricted as to make every function of the family (2) admissible (i. e., (i) $\eta$ is of class $C'$, (ii) $\eta(x_1) = \eta(x_2) = 0$, (iii) $\epsilon$ sufficiently small).* The integral determined for the functions (2) will then become a function of $\epsilon$,

$$I(\epsilon) = \int_{x_1}^{x_2} f(x_1, y_0 + \epsilon\eta, y_0' + \epsilon\eta')dx, \tag{3}$$

which must possess a minimum for $\epsilon = 0$. Hence we obtain, as necessary conditions,

$$I'(0) = 0, \qquad I''(0) \geqq 0, \ (\leqq 0). \tag{4}$$

Conditions on the function $y_0(x)$ derived from $(4_1)$ are called *first-order conditions*; those derived from $(4_2)$ are called *second-order conditions*. It is readily found† that

$$I'(0) = \int_{x_1}^{x_2} (f_y\eta + f_{y'}\eta')dx, \tag{5}$$

in which $f_y = \partial f/\partial y$, $f_{y'} = \partial f/\partial y'$ and in which the arguments of $f_y$ and of $f_{y'}$ are $x$, $y_0(x)$, $y_0'(x)$. The second term in (5) is now integrated by parts; thus we obtain by making use of condition (ii) on the admissible variation $\eta$, the condition that the function $y_0(x)$ must be such that

$$\int_{x_1}^{x_2} \eta(f_y - \frac{d}{dx} f_{y'})dx = 0, \tag{6}$$

*for every admissible variation* $\eta$.

At this stage enters the *fundamental lemma of the calculus of variations*:

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* Functions $\eta(x)$ which satisfy conditions (i) and (ii) are called admissible variations.

† See, e. g., Bolza, *Lehrbuch der Variationsrechnung*, p. 21.

FUNDAMENTAL LEMMA. *If $M$ is a continuous function of $x$, and if for every function $\eta$ of class $C'$ for which $\eta(x_1) = \eta(x_2) = 0$, we have $\int_{x_1}^{x_2} \eta M dx = 0$, then $M \equiv 0$ on $(x_1, x_2)$.*

This lemma, which appears to have been proved for the first time by Stegemann in 1854, and the need for which does not seem to have been felt by earlier writers, is then applied to equation (6). Thus is obtained the *first necessary condition*, which a solution of the problem must satisfy, viz.,

$$\text{(I)} \qquad f_y(x, y_0, y_0') - \frac{d}{dx} f_{y'}(x, y_0, y_0') = 0.$$

This equation is known as the *Euler equation*. A function which satisfies it is called an *extremal*.

2. *Second order conditions* Proceeding now to condition $(4_2)$, we shall consider only the conditions for a *minimum*.* It is then found that the condition

$$\text{(7)} \qquad \int_{x_1}^{x_2} (f_{yy}\eta^2 + 2f_{yy'}\eta\eta' + f_{y'y'}\eta'^2) dx \geqq 0$$

must hold for every admissible variation, the arguments of $f_{yy}$, etc., being $x, y_0(x)$, and $y_0'(x)$. The classical methods for the further study of this condition are based upon a transformation of the integrand which shall make it possible to state conditions which will insure for it constancy of sign for all admissible variations $\eta$. We must refer the reader to the literature† for an account of the way in which (7) leads to the following two conditions:

$$\text{(II)} \qquad f_{y'y'}(x, y_0(x), y_0'(x)) \geqq 0, \qquad x_1 \leqq x \leqq x_2,$$

(*Legendre's condition*);

* Throughout this paper the minimum problems only will be considered; obvious modifications are necessary for the case of a maximum.

† See, e.g., Bolza, loc. cit., Chap. II; Kneser, *Lehrbuch der Variationsrechnung*, 2d ed., §21; Hadamard, *Leçons sur le Calcul des Variations*, p. 313–359.

(III) $$x_2 \leqq x_1',$$

(*Jacobi's condition*).

Here $x_1'$ designates the first vanishing point after $x_1$ of a solution of the ordinary linear differential equation

$$(8) \qquad \left(f_{vv} - \frac{df_{vv'}}{dx}\right)u - \frac{d}{dx}(f_{v'v'}u') = 0,$$

which vanishes at $x_1$. Equation (8) in which $f_{vv}$, $f_{vv'}$ and $f_{v'v'}$ have the arguments $x$, $y_0(x)$, $y_0'(x)$ and are therefore functions of $x$, is known as *Jacobi's differential equation.* The condition (III) can be stated in different ways. Leaving aside the purely geometrical formulation, due to Kneser,* we shall state condition (III) in a form which introduces the important concept of a *field of extremals.* A *field* of extremals is a region $S$ of the $xy$-plane such that through every one of its points (with the possible exception of a finite number of points) there passes a *unique* extremal. With this definition we can now replace (III) by the equivalent condition

(III'): $y=y_0(x)$, $x_1 \leqq x \leqq x_2$, must be such that there exists a field of extremals $S$ which contains the curve determined by $y=y_0(x)$, $x_1 \leqq x \leqq x_2$, in its interior.

The slope of the unique extremal which passes through the point $(x, y)$ of a field $S$ is denoted by the symbol $p(x, y)$ called the *slope function* of the field. If a one-parameter family of extremals $y=\varphi(x, a)$, $a_1 \leqq a \leqq a_2$, furnishes a field $S$, it follows that the defining equation can be solved for $a$, whenever $(x, y)$ is in $S$; the condition is usually imposed that this function $a(x, y)$ shall be of class $C'$ in $S$. In this case it follows that the function $p(x, y)$ is given by the formula

$$p(x, y) = \varphi_x(x, a(x, y))$$

and that this function is then also of class $C'$.

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* See Bolza, loc. cit., § 13; Bliss, *Calculus of Variations*, p. 140 et seq.; Kneser, loc. cit., § 19.

3. *Weierstrass Condition.* Weierstrass showed that conditions (I), (II), (III) are not sufficient for a solution of the problem and obtained a further necessary condition. It will suit our purpose best to relate this condition to other results which were secured later.

Hilbert showed* that the integral

$$(9)\quad I^{+} = \int_a^b \{f(x,y,p(x,y)) + (y' - p(x,y))f_{y'}(x,y,p(x,y))\}\,dx$$

taken along curves which lie entirely in a field $S$ is *independent of the path.* It should be noticed that the integral (9) taken along an extremal is identical with the integral (1) along that extremal.

From this fact it now follows readily that if an extremal $C_0[y=y_0(x)]$ and an arbitrary admissible curve $C[y=y(x)]$ both lie in a field $S$, then

$$(10)\quad \Delta I = I_C - I_{C_0} = \int_{x_1}^{x_2} E(x,y(x),p(x,y(x)),y'(x))\,dx,$$

in which $E$ is a function of four arguments introduced by Weierstrass and related to the function $f$ of the problem by the formula

$$(11)\quad E(x,y,u,v) \equiv f(x,y,v) - f(x,y,u) - (v-u)f_{y'}(x,y,u).$$

Formula (10), which expresses the Weierstrass Theorem, leads now to the *Weierstrass necessary condition.*†

(IV): *If* $y=y_0(x)$, $x_1 \leqq x \leqq x_2$ *is a solution of our problem, then we must have* $E(x, y_0, y_0', p) \geqq 0$, *for every finite value of* $p$.

It should be observed here that condition (IV) is not derived from $(4_2)$; it is in fact entirely independent of the mode of reasoning upon which conditions (4) were based. Bolza‡ has shown that conditions (I), (II), (III), and (IV)

* See Bolza, loc. cit., p. 108.
† Bolza, loc. cit., §18.
‡ Loc. cit., pp. 116–119.

are not sufficient, and he derived a fifth necessary condition; this we shall not discuss here.

The theory proceeds at this point to set up sufficient conditions, chiefly by means of Weierstrass's theorem as expressed by (10). An important distinction should be introduced in the formulation of these conditions.

A solution of the problem stated in the beginning is said to furnish a *strong* extremum for the integral (1). If the admissible functions $Y(x)$ for which the value of the integral is compared with that along a solution $y_0(x)$ are restricted not merely by the condition

$$|Y(x) - y_0(x)| \leqq d, \tag{12}$$

but also by

$$|Y'(x) - y_0'(x)| \leqq d,$$

the solution is said to furnish a *weak* minimum. Sets of necessary and of sufficient conditions can now be formulated for a strong and for a weak minimum.*

4. *Various Extensions.* We consider now some modifications of the problem.

(a) The condition (2) on admissible functions may be replaced by the condition (2a), that the point $(x_1, y(x_1))$, or the point $(x_2, y(x_2))$, or both, lie on given curves. In this case we obtain as an additional first-order condition the requirement that the equation

$$f(x, y, y') + (s - y')f_{y'}(x, y, y') = 0$$

hold at the point in which the extremal meets the given curve, where $s$ denotes the slope of the given curve at this point. This condition is known as the *transversality condition*; for the case that $f(x, y, y') \equiv g(x, y)\sqrt{1+y'^2}$, transversality reduces to orthogonality.

(b) If condition (1) on admissible functions be replaced by the condition (1a) that these functions be of class $D'$, we obtain the first-order condition that at every point where a discontinuity in the derivative occurs (at such points

* See, e. g., Bolza, loc. cit., p. 127; Hadamard, loc. cit., p. 389, p. 397.

progressive and regressive derivatives $y'^+$ and $y'^-$ will exist), the following equations must hold:

$f_{y'}(x, y, y'^+) = f_{y'}(x, y, y'^-)$, and $f - y'f_{y'}|^+ = f - y'f_{y'}|^-$.

This condition, known as the corner-point condition, is due to Erdmann (1877).*

(c) Weierstrass considered the problem of determining a curve, $x=x(t)$, $y=y(t)$, $t_1 \leqq t \leqq t_2$, which will minimize the integral

$$\int_{t_1}^{t_2} F(x, y, x', y')dt, \tag{13}$$

where now the symbol $'$ denotes differentiation with respect to the parameter $t$. Leaving it to the reader to set up for the curve-problem an exact formulation analogous to the one given above for the function-problem, we may say that the Weierstrass form of the simplest problem of the calculus of variations (usually referred to as the parametric form) consists in the determination of a curve $x=x(t)$, $y=y(t)$, $t_1 \leqq t \leqq t_2$, which will furnish the extreme values of the integral (13). Conditions analogous to those mentioned for the function problem are arrived at in this case and we shall refer to these conditions freely, without explicitly stating them. It must be observed, however, that if this problem is to have any meaning at all, the integrand $F$ must be such that the integral (13) shall depend on the *curve* only, and shall be independent of the particular parameter used for its representation by means of the function $x(t)$ and $y(t)$. This leads to the homogeneity condition,† according to which we must have

$$F(x, y, kx', ky') \equiv kF(x, y, x', y'), \tag{14}$$

whenever $k>0$. From this condition on the function $F$ follow important consequences, among which we mention that there must exist a function $F_1$ such that

$$F_{x'x'} = y'^2F_1, \quad F_{x'y'} = -x'y'F_1, \quad F_{y'y'} = x'^2F_1.$$

* See, e. g., Bolza, loc. cit., p. 367.

† See, Bolza, loc. cit., p. 193–195; Kneser, loc. cit., p. 11.

This function $F_1$ plays a role in the curve problem analogous to that played by $f_{y'y'}$ in the function problem.

5. *Recent Developments.* We are in a position now to mention some of the recent advances made in connection with the theory of the simplest problem.

(a) It was observed by Dubois-Reymond* in 1879 that the fundamental lemma of the calculus of variations (see §1, above) as applied to the integral (6), presupposes that $y''(x)$ exists and is continuous, which was not among the hypotheses laid down for the admissible curves. To meet the difficulty suggested by this remark, he proved the following generalization of the fundamental lemma. *If $M$ is continuous and if for every function $\eta$ of class $C'$ for which $\eta(x_1)=\eta(x_2)=0$, we have $\int_{x_1}^{x_2}\eta' M\,dx=0$, then $M$ is constant throughout $(x_1, x_2)$.*

To apply this lemma, he integrated the first term in (5), so as to derive from $(4_1)$ the condition that

$$\int_{x_1}^{x_2}\eta'\left(f_{y'}-\int_{x_1}^{x} f_y\,dx\right)dx=0, \tag{15}$$

for every admissible variation $\eta$; the Lemma of Dubois-Reymond then gives the condition

$$f_{y'}-\int_{x_1}^{x} f_y\,dx=\text{const.} \tag{16}$$

The conditions on the admissible functions and on the function $f$ are sufficient to enable us to conclude from this that the derivative of $f_{y'}$ exists and is continuous on $(x_1, x_2)$, and thus to obtain the Euler equation (I). It follows from this work that an admissible function which solves our problem must be of class $C''$.

Various further generalizations of the fundamental lemma in the direction suggested by Dubois-Reymond have since been made by others.† The most inclusive generalization

* See Bolza, loc. cit., p. 27.

† See Zermelo, MATHEMATISCHE ANNALEN, vol. 59 (1904), p. 558; Jacobstahl, ARCHIV DER MATHEMATIK UND PHYSIK, vol. 16 (1910), p. 82.

has been made recently by A. Haar,* who showed that if $v(x)$ is continuous, and such that for every function $u(x)$ of class $C^{(k)}$ for which $u^{(i)}(a)=u^{(i)}(b)=0$, $(i=0, \cdots, k-1)$, we have $\int_a^b vL(u)dx=0$, $L$ being a linear homogeneous differential form of $k$th order whose coefficients are functions of $x$ possessing derivatives of all orders and whose leading coefficient is unity, then $v(x)$ will be of class $C^{(k)}$ on $(ab)$ and $\Lambda(v)=0$, where $\Lambda$ is the adjoint of $L$. The proof is simple enough to be substituted for the usual proof of the fundamental lemma and of Dubois-Reymond's Lemma; these it contains as special cases for $L(u)=u$ and $L(u)=u'$ respectively.

Another way of dealing with the difficulty pointed out by Dubois-Reymond has been suggested by Razmadze,† who proved the following theorem: *If $M(x)$ and $N(x)$ are continuous on* (a, b), *and if for every function $\eta(x)$ of class $C'$ such that $\eta(a)=\eta(b)=0$, we have $\int_a^b(M\eta+N\eta')dx=0$, then $N'$ exists at every interior point of* $(a, b)$, *and* $N'=M$. This theorem enables us to pass directly from equation (5) to the Euler equation, without integration by parts either of the second term or of the first term. This result of Razmadze has been further extended in an interesting paper by Kryloff‡ in such a way as to provide also for the case in which higher derivatives are involved. He proved that, if for every function $\eta$ of class $C^{(n)}$ for which

$$\eta^{(i)}(a)=\eta^{(i)}(b)=0, \quad i=1, \cdots, n-1,$$

we have

$$\int_a^b \sum_{i=0}^{n} M_i\eta^{(i)}dx = 0, \tag{17}$$

then we can conclude that

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* See ACTA LITTERARUM AC SCIENTIARUM R. U. H. FRANCIS-JOSEPHINAE, vol. 1 (1922), p. 33.

† See MATHEMATISCHE ANNALEN, vol. 84 (1921), p. 115.

‡ See BULLETIN DE L'ACADÉMIE DES SCIENCES DE L'OUARAÏNE (Classe des sciences physiques et mathématiques), vol. 1 (1923), p. 8.

$$\frac{d}{dx}\left\{\frac{d}{dx}\cdots\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{d}{dx}M_n - M_{n-1}\right) + M_{n-2}\right]\right.$$
$$\left. - M_{n-3} + \cdots + (-1)^{n-1}M_1\right\} + (-1)^n M_0 = 0,$$

provided the functions $M_i$ are such that the successive derivatives which appear in this formula exist. If we suppose that the functions $M_i$ are of class $C^{(i)}$, then the last conclusion may be written in the form

$$\sum_{i=0}^{n}(-1)^i M_i^{(i)} = 0.$$

It is this result which I have had in my possession for a number of years, with the intention of using it in the study of a generalization of the simplest problem to which reference will be made in the sequel. In this result, the equation (17) is supposed to hold for every function $\eta$ of class $C^{(n)}$ which satisfies two general linear homogeneous boundary conditions of the form

$$\sum_{j=0}^{n}\alpha_{ij}\eta^{(j)}(a) + \beta_{ij}\eta^{(j)}(b) = 0, \qquad (i = 1, \cdots, 2n),$$

$\alpha_{ij}$ and $\beta_{ij}$ being constants.

(b) We have seen that conditions (II) and (III) are obtained in the classical theory of the simplest problem from the condition $(4_2)$. It has also been mentioned that Kneser has developed a geometrical formulation of (III). In the study of more complicated problems, and even in connection with the simplest problem in parametric form, it was found that the transformation of the second variation, analogous to the one referred to under §2, became very complicated. The tendency has been therefore to follow Kneser's geometric method, which indeed is capable of extension to such problems in a very elegant manner. This, however, has the drawback of leaving unsettled exceptional cases, arising out of possible singularities of the curves involved. To meet the difficulty which arises in this way, Bliss has developed within recent years a new method of obtaining the second-order conditions from the

second variation. This method has been applied by him and by several of his pupils to a number of problems.* It is easy to explain its essential features in connection with the problem with which we have thus far been concerned. The condition $(4_2)$ has been said to lead to condition (7). If we denote the integrand in (7) by $2\Omega(\eta, \eta')$, i.e.,

$$2\Omega(\eta, \eta') \equiv f_{yy}\eta^2 + 2f_{yy'}\eta\eta' + f_{y'y'}\eta'^2, \tag{18}$$

condition (7) may be interpreted as requiring that 0 shall be the minimum value of the integral $\int_{x_1}^{x_2} \Omega(\eta, \eta')dx$. Hence any admissible variation $\eta(x)$ which gives this latter integral the value 0 must satisfy the conditions necessary for an extremal of this integral, looked upon as an integral in the $x\eta$-space. By applying to the integral of (18) the necessary conditions for a minimum which do not depend on $(4_2)$, the second-order conditions may be obtained. It is seen that the Euler equation obtained from the function (18) is identical with the Jacobi equation (8). By applying the Weierstrass condition to the function $\eta \equiv 0$, which gives the integral of (18) its minimum value and hence must satisfy the necessary conditions, one obtains the Legendre condition for the original integral. Bliss shows furthermore that if the Jacobi equation (8) possesses an integral which vanishes at $x_1$ and again at a point $x_1' < x_2$, then it is possible to construct a function of class $D'$ which gives the integral $\int \Omega dx$ its minimum value 0, but which fails to satisfy the corner-point condition; it would therefore be possible to render this integral negative. Thus the necessity of condition (III) is established.

(c) We turn next to another extension of the simplest problem. In a recent paper, Razmadze† has investigated

* See Bliss, this BULLETIN, vol. 26 (1920), p. 343, and TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), p. 195. Also D. M. Smith, TRANSACTIONS, vol. 17 (1916), p. 459; M. B. White, TRANSACTIONS, vol. 13 (1912), p. 175; G. A. Larew, TRANSACTIONS, vol. 20 (1919), p. 1; F. LeStourgeon, TRANSACTIONS, vol. 21 (1920), p. 357.

† See MATHEMATISCHE ANNALEN, vol. 94 (1925), p. 1.

the conditions which must be satisfied by a *discontinuous curve* which furnishes a minimum for the integral (1). He supposes that a solution for this problem is given by a curve which is of class $C'$ throughout the interval $(x_1, x_2)$ except at one point $x_0$, where it has a finite discontinuity.

The importance of this question becomes clear when one considers the problem of Weierstrass, in which it is required to minimize the integral $\int_{-1}^{+1} x^2y'^2dx$ by a function $y(x)$, for which $y(-1)=a$, $y(1)=b$, $a\neq b$.* The Euler equation for this problem becomes $x^2y'=\text{const.}$ which has, besides the singular integral $x=0$, the general integral $y=c_1/x+c_2$. Hence it is clear that there exists no continuous extremal joining the given points. On the other hand, it is readily seen that the greatest lower bound of the values of the integral for the continuous functions joining the given points is 0, when one considers the family of functions $C_\epsilon$ given by the equation

$$y = \frac{a+b}{2} + \frac{(b-a)}{2} \frac{\arctan(x/\epsilon)}{\arctan(1/\epsilon)},$$

for which the value of the integral becomes

$$J(\epsilon) < \frac{\epsilon(b-a)^2}{2\arctan(1/\epsilon)},$$

so that $J(\epsilon)\to 0$, when $\epsilon\to 0$. When $\epsilon\to 0$, the function $C_\epsilon$ tends toward the discontinuous function given by the equations

$$\begin{aligned} y &= a \quad \text{for} \quad -1 \leqq x < 0, \\ y &= (a+b)/2 \quad \text{for} \quad x = 0, \\ y &= b \quad \text{for} \quad 0 < x \leqq 1; \end{aligned}$$

for this the integral does indeed take its minimum value 0.

What are the conditions which a curve with a finite discontinuity must satisfy if it is to minimize the integral (1)? The first-order conditions are found to be, besides the

* See Bolza, loc. cit., p. 420.

requirement that the continuous parts of the curve should be extremals, the equations

$$f(x_0, y_0, y_0') = f(x_0, \bar{y}_0, \bar{y}_0'), \quad f_{y'}(x_0, y_0, y_0') = f_{\bar{y}'}(x_0, \bar{y}_0, \bar{y}_0') = 0,$$

where $(x_0, y_0)$ and $(x_0, \bar{y}_0)$ are the two points between which the finite discontinuity takes place, and $y_0'$ and $\bar{y}_0'$ are the slopes of the parts of the curve at these points. These conditions are necessary in the general case, in which discontinuous comparison curves can be admitted whose break *need not* occur for the same value of $x$ as gives the cut-point for the solution. In the exceptional case in which the cut-points of all admissible discontinuous comparison curves have the same abscissa, the first of the above conditions is replaced by a condition which determines the abscissa of the cut-point. A further necessary condition is that if $y_k$ lies between $y_0$ and $\bar{y}_0$, and $y_k'$ is arbitrary, we must have $f(x_0, y_k, y_k') \geqq f(x_0, y_0, y_0')$ and $f(x_0, y_k, y_k') \geqq f(x_0, \bar{y}_0, \bar{y}_0')$.

Assuming further that the two continuous parts of the curve satisfy the conditions of Legendre, Jacobi, and Weierstrass for continuous solutions of the problem, Razmadze obtains a theory of conjugate points, further necessary conditions, and also sufficient conditions. For the detailed results the reader is referred to Razmadze's paper.

(d) In 1907, Bliss introduced a new form for the simplest problem of the calculus of variations, by asking for a curve such that

$$\int_{t_1}^{t_2} F(x, y, \theta)\, dt$$

becomes a minimum, in which $\theta$ is defined by the equations*

$$\cos\theta = \frac{x'}{\sqrt{x'^2 + y'^2}}, \quad \text{and} \quad \sin\theta = \frac{y'}{\sqrt{x'^2 + y'^2}}.$$

The theory is in some respects simpler than in the ordinary Weierstrass formulation, inasmuch as the quantities $x$, $y$, $\theta$ are independent of the choice of parameter.

* See TRANSACTIONS OF THIS SOCIETY, vol. 8 (1907), p. 405.

This method of treatment has since been extended to spaces of three and more dimensions by Rider* and Sakellariou,† who have also developed the theory of the Hilbert independent integral and the Weierstrass $E$-function for integrals of such form.

(e) For some years I have had under consideration a modification of the simplest problem of the calculus of variations, obtained by the consideration of more general boundary conditions. The problem may roughly be formulated as follows.

To minimize the integral (1) by a function which satisfies two linear boundary conditions of the form

$$\alpha_{i1}y(a) + \alpha_{i2}y(b) + \beta_{i1}y'(a) + \beta_{i2}y'(b) = C_i, \quad i = 1,2.$$

It is for the purpose of this problem that the extension of the fundamental lemma mentioned in §5(a) was developed.

## II. The General Lagrange Problem and the Mayer Problem

1. *Introduction.* A variety of problems of the calculus of variations is included in the *Lagrange problem*, which is concerned with an integral of the form

$$\int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx, \tag{19}$$

where $f$ is a function of class $C'''$ with respect to its $2n+1$ arguments in a region $R_1$ of $(2n+1)$-space, determined by the conditions that $(x, y_1, \cdots, y_n)$ be in a certain region $R$ of $(n+1)$-space and that‡ $y_i'$, $i=1, \cdots, n$ be finite. The problem consists in determining among all sets of $n$ functions

* See Tôhoku Mathematical Journal, vol. 13 (1918), p. 165; Washington University Studies, vol. 5 (1918), p. 97; American Journal of Mathematics, vol. 39 (1917), p. 241.

† See Tôhoku Mathematical Journal, vol. 13 (1918), p. 15; Annali di Matematica, vol. 28 (1919), p. 169; Palermo Rendiconti, vol. 44 (1920), p. 53.

‡ The indices $i$, $j$, and $k$ will be understood to run over the range $1, \cdots, n$ throughout the sequel, unless differently indicated.

$y_i(x)$, $x_1 \leqq x \leqq x_2$, which (1) consist of functions of class $C'$, (2) satisfy the conditions $y_i(x_1) = y_{i1}$, $y_i(x_2) = y_{i2}$, (3) are such that the points $(x, y_1, \cdots, y_n)$ for $x_1 \leqq x \leqq x_2$ lie in the region $R$, (4) satisfy the differential equations.

$$\varphi_\mu(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx = 0, \tag{20}$$

for $\mu = 1, \cdots, m < n$, a set of functions $y_{i0}(x)$ for which there exists a positive number $d$, such that for all sets of functions $Y_i(x)$, $x_1 \leqq x \leqq x_2$, which satisfy conditions (1) to (4), and the further conditions that $|Y_i(x) - y_{i0}(x)| < d$ for $x_1 \leqq x \leqq x_2$, the integral (19) has a value not less than (not greater than) the value which it has for $y_{i0}(x)$. A set of functions which fulfils conditions (1) to (4) is said to determine an *admissible curve in* $(n+1)$-space for this problem.

A second general formulation which includes a large number of problems of the calculus of variations is given by the *problem of Mayer*, which may be stated as follows. To determine among all sets of functions $y_i(x)$, $x_1 \leqq x \leqq x_2$, which (1) consist of functions of class $C'$; (2) satisfy the conditions $y_i(x_1) = y_{i1}$, $y_\nu(x_2) = y_{\nu 2}$, $\nu = 2, \cdots, n$; (3) are such that the points $(x, y_1, \cdots, y_n)$ for $x_1 \leqq x \leqq x_2$ lie in a given region $R$ of the $xy$-plane; (4) satisfy the differential equations $\varphi_\mu(x, y_1, \cdots, y_n; y_1', \cdots, y_n') = 0$, $\mu = 1, \cdots, m < n$, a set of $n$ functions $y_{i0}(x)$ for which there exists a positive number $d$, such that, if $Y_i(x)$ be any other set of functions, which satisfies conditions (1) to (4), and the further condition that $|Y_i(x) - y_{i0}(x)| < d$ for $x_1 \leqq x \leqq x_2$, then $Y_i(x_2)$ shall be not less than (not greater than) $y_{i0}(x_2)$. The theory of these problems, their interrelations and their importance in the study of general problems of the calculus of variations have formed the subject of many studies.* The first step in those studies consists in establishing a problem in which the admissible curves are restricted by conditions (1), (2), and (3) only, and which is, at least as far as first-

*See, e.g., Bolza, loc. cit., p. 543 and p. 573; Hadamard, loc. cit., p. 217–224, p. 176; Kneser, loc. cit., Chap. 6.

order conditions are concerned, equivalent to the given problem. This is accomplished by means of the *Euler-Lagrange multiplier rule*,* which for the Lagrange problem may be stated as follows. If $y_i = y_{i0}(x)$ furnish a solution for the general Lagrange problem, then there must exist $m$ functions $\lambda_\mu(x)$ such that $y_{i0}$ and $\lambda_\mu$ satisfy the first order conditions for a minimum of the integral

$$\int_{x_1}^{x_2} F(x, y_i, y_i', \lambda_\mu) dx, \tag{21}$$

the class of admissible functions being determined by conditions (1), (2), (3), and where

$$F = f + \sum \lambda_\mu \varphi_\mu. \tag{22}$$

For the Mayer problem a similar theorem has been proved. This multiplier-rule has been established also for problems in which to the condition (4) is adjoined a condition requiring that the unknown functions $y_i(x)$ also satisfy a number of finite equations. Difficulties of a particular kind, clearly recognized by Hahn,† had to be overcome in such cases. Other difficulties arise when the conditions (2) are replaced by more general conditions as to the end-points. These circumstances have given rise to a thorough working over of the subject, in the course of which further extensions and other valuable results have been secured. Mention must be made of a very elegant and complete treatment of this subject given by Bliss in a course of lectures at the University of Chicago, during the summer of 1925.‡

2. *Extensions and Generalizations.* We proceed to report briefly on some of the recent treatments of the Euler-Lagrange rule.

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* See Bolza, loc. cit., § 66–72.

† MATHEMATISCHE ANNALEN, vol. 58 (1903), p. 152; see also Bolza, loc. cit., p. 564, where further references are given.

‡ I acknowledge with thanks the opportunity which Professor Bliss has given me to see a mimeographed copy of these lectures, prepared by Mr. O. E. Brown, Northwestern University.

(a) In the second of two papers, devoted to this subject, Bolza* has considered the following parametric problem.† To minimize the expression

$$U = \int_{t_0}^{t_1} f(y_1, \cdots, y_n, y_1', \cdots, y_n')dt + G(y_{10}, \cdots, y_{n0}, y_{11}, \cdots, y_{n1}),$$

when the admissible curves are determined by functions

$$y_i = y_i(t), \qquad t_0 \leqq t \leqq t_1,$$

which satisfy the differential equations

$$\varphi_\alpha(y_i, y_i') = 0, \qquad \alpha = 1, \cdots, p,$$

and the finite equations

$$\psi_\beta(y_i) = 0, \qquad \beta = 1, \cdots, q,$$

while the end-points $y_{i0}$, $y_{i1}$ satisfy the equations

$$\chi_\gamma(y_{i0}, y_{i1}) = 0, \qquad \gamma = 1, \cdots, r;$$

it is moreover stipulated that along the minimizing curve $y_i = y_{i0}(x)$, the determinant

$$(23) \qquad \begin{vmatrix} \dfrac{\partial \varphi_\alpha}{\partial y_1'}, & \cdots, & \dfrac{\partial \varphi_\alpha}{\partial y_m'} \\ \dfrac{\partial \psi_\beta}{\partial y_1}, & \cdots, & \dfrac{\partial \psi_\beta}{dy_m} \end{vmatrix} \neq 0, \qquad m = p + q < n,$$

and also that the matrix

$$(24) \qquad \left\| \begin{matrix} \dfrac{\partial \psi_\beta}{\partial y_1}\Big|_{y_{i0}}, & \cdots, & \dfrac{\partial \psi_\beta}{\partial y_n}\Big|_{y_{i0}}, & 0, \cdots, & 0 \\ 0, & \cdots, & 0, & \dfrac{\partial \psi_\beta}{\partial y_1}\Big|_{y_{i1}}, & \cdots, & \dfrac{\partial \psi_\beta}{\partial y_n}\Big|_{y_{i1}} \\ \dfrac{\partial \chi_\gamma}{\partial y_{10}}, & \cdots, & \dfrac{\partial \chi_\gamma}{\partial y_{n0}}, & \dfrac{\partial \chi_\gamma}{\partial y_{11}}, & \cdots, & \dfrac{\partial \chi_\gamma}{\partial y_{n1}} \end{matrix} \right\|,$$

* See MATHEMATISCHE ANNALEN, vol. 64 (1907), p. 370; ibid., vol. 74 (1913), p. 430; see also Hilbert, MATHEMATISCHE ANNALEN, vol. 62 (1906), p. 351.

† To save space, I shall only give a rough statement of the different problems that are discussed; by comparing with the more complete statements given on pages 475 and 484, the reader will have little difficulty in supplying exact formulations.

of $2q+r$ rows and $2n$ columns shall be of rank $2q+r$ for the arguments furnished by the end-points of the minimizing curve.

It is clear that this problem reduces to the most general Lagrange problem with general boundary conditions, if $G\equiv 0$, and to the Mayer problem with general boundary conditions if $f\equiv\psi_\beta\equiv 0$, and $G\equiv y_{11}$.

Bolza proved that if $y_i=Y_i(t)$, $i=1, \cdots, n$, furnishes a solution for this problem, then there exist $p+q$ functions $\lambda_\alpha(t)$ and $\mu_\beta(t)$, and also $2q+r+1$ constants $l_0$, $l_\beta^0$, $l_\beta^1$, $l_{q+\gamma}$ such that the functions $Y_i$, $\lambda_\alpha$ and $\mu_\beta$ satisfy the system of $n$ differential equations

$$\frac{\partial\Omega}{\partial y_i}-\frac{d}{dt}\frac{\partial\Omega}{\partial y_i'}=0,$$

and the $2n$ boundary conditions

$$H_i^0=0, \quad H_i^1=0.$$

Here

$$\Omega=l_0 f+\sum\lambda_\alpha\varphi_\alpha+\sum\mu_\beta\psi_\beta,$$

and

$$H_i^\epsilon=(-1)^{\epsilon+1}\frac{\partial\Omega}{\partial y_i'}\Big|_{t_\epsilon}+l_0\frac{\partial G}{\partial y_{i\epsilon}}+\sum l_\beta^\epsilon\frac{\partial\psi_\beta}{\partial y_i}\Big|_{t_\epsilon}+\sum l_{q+\gamma}\frac{\partial\chi_\gamma}{\partial y_{i\epsilon}}, \qquad \epsilon=0,1,$$

and the constants $l_0$, $l_{q+\gamma}$ and $l_\beta=l_\beta^0+l_\beta^1+\int_{t_0}^{t_1}\mu_\beta\,dt$ do not all vanish.

This result holds both for the *normal* and for the *abnormal* cases; these are obtained for $l_0\neq 0$ and $l_0=0$, respectively. Furthermore, the usual theorems as to uniqueness of the constant and of the functional multipliers are established for the normal case; and the abnormal case receives a simple characterization. The difficulties hinted at as arising from the finite conditions $\psi_\beta(y_i)=0$ are obviated by means of the

condition on the matrix (24), which insures against redundancy in the conditions arising from the finite equations and from the end-point equations.

(b) Another general form of the Euler-Lagrange rule has been given by Bliss,* who treated the following problem:

To determine a set of functions of class $C''$, $y_i = y_i(x)$, $x_1 \leqq x \leqq x_2$, which satisfy the differential equations (20), and the boundary conditions

$$f_{\gamma'}(x_1, y_{i1}, x_2, y_{i2}) = 0, \qquad \gamma' = 2, \cdots, r,$$

while the function $f_1(x_1, y_{i1}, x_2, y_{i2})$ is minimized.

It is an important feature of this formulation that the condition (23) is replaced by the less restrictive one that the matrix $||\partial\varphi_\mu/\partial y_i'||$ shall everywhere be of rank $m$. By proving that a matrix of $mn$ continuous functions and of rank $m$ at every point of $(x_1, x_2)$, can always be extended into a determinant of order $n$, which vanishes nowhere on $(x_1, x_2)$, Bliss succeeds in accomplishing with this more general condition the same purpose for which a condition like (23) had always been introduced, viz., that of setting up a family of admissible comparison curves. The further requirement is made that the matrix

$$\left\| \frac{df_\gamma}{\partial x_1} \quad \frac{df_\gamma}{dy_{i1}} \quad \frac{df_\gamma}{dx_2} \quad \frac{df_\gamma}{dy_{i2}} \right\|, \qquad \gamma = 1, \cdots, r,$$

of $r$ rows and $2n+2$ columns shall be of rank $r$ at the end-points of the minimizing curve; this is seen to be analogous to the earlier condition on the matrix (24). This general problem, like the one mentioned under (a), includes most of the problems of the calculus of variations, which are concerned with simple integrals, and the interrelations are pointed out, although not completely. For the present problem, the following result is obtained:

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* See TRANSACTIONS OF THIS SOCIETY, vol. 19 (1918), p. 305.

If the functions $Y_i(x)$ furnish a solution, then there exist $m$ functions $\lambda_\mu$ of class $C'$, not all identically 0 on $(x_1, x_2)$, such that the functions $Y_i$ and $\lambda_\mu$ satisfy the differential equations

$$\frac{\partial \Omega}{\partial y_i} - \frac{d}{dx}\frac{\partial \Omega}{\partial y_i'} = 0,$$

and the boundary conditions, which are expressed by the requirement that the rank of the matrix

$$\left\| \begin{matrix} \dfrac{\partial f_\gamma}{\partial x_1}, & \dfrac{\partial f_\gamma}{\partial y_{i1}}, & \dfrac{\partial f_\gamma}{\partial x_2}, & \dfrac{\partial f_\gamma}{\partial y_{i2}} \\ \Omega(x_1) - \Upsilon_1, & \dfrac{\partial \Omega}{\partial y_i'}\Big|_{x_1}, & -\Omega(x_2) + \Upsilon_2, & -\dfrac{\partial \Omega}{\partial y_i'}\Big|_{x_2} \end{matrix} \right\|$$

of $r+1$ rows and $2n+2$ columns shall be less than $r+1$; here $\Omega = \Sigma \lambda_\mu \varphi_\mu$, and

$$\Upsilon_1 = \sum y_i' \left.\frac{\partial \Omega}{\partial y_i'}\right|_{x_1}, \qquad \Upsilon_2 = \sum y_i' \left.\frac{\partial \Omega}{\partial y_i}\right|_{x_2}.$$

(c) An apparently very wide generalization of the Euler-Lagrange rule has recently been given by Hahn,* who considers the following problem. Suppose given $r$ functional operators

$$W_\gamma(y_i, a, b), \qquad \gamma = 1, \cdots, r,$$

each of which associates a real number with every set of $n$ functions $y_i(x)$ of class $C'$, $a \leqq x \leqq b$, which satisfies the conditions

$$(25) \qquad |y_i(x) - y_{i0}(x)| < h, \; |y_i'(x) - y_{i0}'(x)| < h,$$

$$|a - a_0| < h, \; |b - b_0| < h,$$

where $y = y_{i0}(x)$, $a_0 \leqq x \leqq b_0$, is some particular set of functions of class $C'$, and where $h$ is a positive constant. Furthermore, let

$$\Phi_\mu(y_i, a, b, t), \qquad \mu = 1, \cdots, m,$$

* See WIENER BERICHTE, vol. 131 (1922), p. 531.

define a similar operator for every $t$ on $(t_1, t_2)$. It is supposed that

(1) If

$$(26) \qquad y_i(x) = y_{i0}(x) + \sum_{\sigma=1}^{s} \epsilon_\sigma \eta_{i\sigma}(x) + (\epsilon),$$

$$y_i'(x) = y_{i0}'(x) + \sum_{\sigma=1}^{s} \epsilon_\sigma \eta_{i\sigma}'(x) + (\epsilon),$$

$$a = a_0 + \sum_{\sigma=1}^{s} \epsilon_\sigma \alpha_\sigma + (\epsilon),$$

$$b = b_0 + \sum_{\sigma=1}^{s} \epsilon_\sigma \beta_\sigma + (\epsilon),$$

where $(\epsilon)$ represents a function of the parameters $\epsilon_1, \cdots, \epsilon_s$ which tends to zero with these parameters, and where $\eta_{i\sigma}$, $\alpha_\sigma$, and $\beta_\sigma$ are so chosen that the conditions (25) are satisfied, then

$$W_\gamma(y_i, a, b) = W_\gamma(y_{i0}, a_0, b_0) + \sum_\sigma \epsilon_\sigma V_\gamma(\eta_{i\sigma}, \alpha_\sigma, \beta_\sigma) + (\epsilon),$$

and

$$\Phi_\mu(y_i, a, b, t) = \Phi_\mu(y_{i0}, a_0, b_0, t) + \sum_\sigma \epsilon_\sigma \Psi_\mu(\eta_{i\sigma}, \alpha_\sigma, \beta_\sigma, t) + (\epsilon),$$

in which $V_\gamma(\eta_i, \alpha, \beta)$ and $\Psi_\mu(\eta_i, \alpha, \beta, t)$ are linear continuous functional operators* with respect to $\eta_i$ and $\eta_i'$, and linear continuous functions of $\alpha, \beta$ and of $\alpha, \beta, t$, respectively.

(2) If $s$ sets of functions $\eta_{i\sigma}(x)$ of class $C''$ on $(a_0, b_0)$ and $s$ sets of constants $\alpha_\sigma, \beta_\sigma$ are given which satisfy the linear functional equations $\Psi_\mu(\eta_i, \alpha, \beta, t) = 0$, then there exists for every set of values of the parameters $\epsilon_1, \cdots, \epsilon_s$, sufficiently small in absolute value, a solution $y_i, a, b$, of the functional equations $\Phi_\mu(y_i, a, b, t) = \Phi_\mu(y_{i0}, a_0, b_0, t)$, of the form (26).

(3) For every $e > 0$, there exists a $d$, such that if $\psi_\mu(t)$ are continuous functions of $t$ on $(t_1, t_2)$, for which $|\psi_\mu(t)| < d$,

* For the definitions of these terms, see, e.g., Lévy, *Leçons d'Analyse Fonctionnelle*, pp. 50, 52.

then there exists at least one set of functions $\eta_i(x)$ of class $C'$ on $(a_0, b_0)$ and a pair of constants $\alpha, \beta$, such that

$$|\eta_i| < e, \qquad |\eta_i'| < e, \qquad |\alpha| < e, \qquad |\beta| < e,$$

and such that

$$\Psi_\mu(\eta, \alpha, \beta, t) \equiv \psi_\mu(t).$$

The following problem is now considered. Among all sets of $n$ functions $y_i(x)$ of class $C'$ which satisfy the conditions (25) and for which

$$W_{\gamma'}(y_i, a, b) = 0, \qquad \gamma' = 2, \cdots, r$$

and

$$\Phi_\mu(y_i, a, b, t) = 0, \qquad \mu = 1, \cdots, m; \qquad t_1 \leqq t \leqq t_2,$$

to determine one which will minimize the operator $W_1(y_i, a, b)$.

It is shown in the first place that there must exist $r$ numbers $l_1, \cdots, l_r$ not all equal to 0, such that

$$V(\eta_i, \alpha, \beta) \equiv \sum l_\gamma V_\gamma(\eta_i, \alpha, \beta) = 0$$

for every solution $\eta_i, \alpha, \beta$ of the linear functional equations

$$\Psi_\mu(\eta, \alpha, \beta, t) = 0.$$

Next it is shown that if

$$\Psi_\mu(\eta_i, \alpha, \beta, t) = \Psi_\mu(\zeta_i, \gamma, \delta, t),$$

then we will also have

$$V(\eta_i, \alpha, \beta) = V(\zeta_i, \gamma, \delta),$$

so that the value of $V(\eta_i, \alpha, \beta)$ depends only upon $\Psi_\mu(\eta, \alpha, \beta, t)$ which are ordinary functions $\psi_\mu$ of $t$. We may then write

$$V(\eta_i, \alpha, \beta) = U(\psi_1, \cdots, \psi_m),$$

where $U$ is a functional operator. This operator is then proved to be linear and continuous. Hence, by the use of a theorem of Riesz,* of which Hahn gives a very elegant

* See, e.g., Lévy, loc. cit., p. 55; also footnote (2) on first page of Hahn's paper mentioned above, in which full references are given for this theorem.

proof in this paper, we conclude that $U$ can be represented as a Stieltjes integral, i.e., that there exist functions of limited variation $v_\mu(t)$, such that

$$U = \sum_{\mu=1}^{m} \int_{t_1}^{t_2} \psi_\mu dv_\mu(t).$$

This result combined with the preceding results, leads to the following theorem.

*If the set of functions $y_i$ furnishes a solution of the problem, then there exist $r$ constants $l_\gamma$ not all 0 and $m$ functions of limited variation $\lambda_\mu$ such that*

$$\sum_{\gamma} l_\gamma V_\gamma(\eta_i, \alpha, \beta) + \int_{t_1}^{t_2} \sum_{\mu=1}^{m} \Psi_\mu(\eta_i, \alpha, \beta, t) d\lambda_\mu(t) = 0$$

*for every set of functions $\eta_i$, of class $C'$ in $(a_0, b_0)$, and for all numbers $\alpha, \beta$.*

This is the Euler-Lagrange rule for this general minimum problem, which includes as special cases the problems mentioned under (a) and (b).

3. *Second Order Conditions.* Other advances made in the theory of the Lagrange problem and of the Mayer problem have to do with second order conditions and with extensions to these problems of the Hilbert independent integral and of the Weierstrass theorem, which were mentioned in connection with the simplest problem of the calculus of variations in I, §3. The first point of importance is that whereas the Hilbert integral, for the simplest problem, is independent of the path of integration in *any* field of extremals, the independence of the analogous integral for these more general problems takes place only in the "Mayer fields," characterized by special conditions. Returning to the Lagrange problem, as formulated in §1, we suppose we have an $n$-parameter family of extremals which furnishes a field, and we define, as in the case of the simplest problem, slope functions of the field $p_i(x, y_1, \cdots, y_n)$, and also functions $q_\mu(x,$

$y_1, \cdots, y_n$) which are obtained as the Lagrange multipliers for the unique extremal which passes through the point. Then it is shown that the integral

$$I^+ = \int_{x_1}^{x_2} \left\{ f(x, y_i, p_i) - \sum_i (p_i - y_i') F_{ui'}(x, y_i, p_i, q_\mu) \right\} dx,$$

in which $F$ is the function defined by (22), will be independent of the path provided the following $n(n-1)/2$ relations are satisfied:

$$\frac{\partial F_{uk'}(x, y_i, p_i, q_\mu)}{\partial y_h} = \frac{\partial F_{uh'}(x, y_i, p_i, q_\mu)}{\partial y_k}, \qquad h, k = 1, \cdots, m.$$

These conditions determine the special fields for which the Hilbert independence property holds and which are known as *Mayer fields*. The construction of Mayer fields, the Weierstrass theorem for Mayer fields, the connection with the transformation of the second variation and with second-order conditions have formed the subject of a number of papers.

(a) The conditions for the independence of the Hilbert integral, both for the case when all curves in a field are considered (absolute invariance) and for the case, suggested by Radon,* in which only curves which satisfy the equations (20) are taken into account (relative invariance), were studied by Bolza,† who also gave a simple geometric interpretation for these conditions, upon which the construction of Mayer fields can be based. He showed that, at least for the special Lagrange problem, to which the problem of minimizing the integral $\int f(x, y, y', \cdots, y^{(n)})$ can be reduced,‡ the relative invariance of the Hilbert integral carries the absolute invariance as a consequence.

The classical treatment of the second variation by Clebsch, Mayer, and von Escherich was based on the

* See WIENER BERICHTE, vol. 119 (1910), p. 1257.

† See PALERMO RENDICONTI, vol. 31 (1911), p. 257, and vol. 32 (1911), p. 111.

‡ See Bolza, *Vorlesungen*, p. 543.

transformation of the second variation to which we have had occasion to refer earlier. The complicated character of these transformations constitutes a serious drawback to this treatment of the second variation. A valuable contribution to the problem of relating these transformations to other parts of the theory was made by Hahn.* A considerable advance was made through the application to the second variation for the integral (19) of the method of Bliss, the essential features of which were discussed in I, §5(b).

(b) This was done by D. M. Smith† who obtained the analogues of the Legendre and Jacobi conditions, by applying the Weierstrass condition and the corner-point conditions to the secondary minimum problem. In a paper presented to the Toronto Congress in 1924, Bliss carried the treatment of the second variation still further by showing how Mayer fields may be constructed for the secondary problem, then using the Hilbert independent integral, and finally deriving the Weierstrass formula for the total variation of the secondary problem in terms of the $E$ function. By taking an arbitrary admissible variation $\eta_i$ for one curve and the curve $\eta_i \equiv 0$ for the other, an expression for the second variation is then obtained, in a very simple manner, which turns out to be exactly the one sought for by the complicated transformations of the earlier theories.

The same form of treatment was carried through for the Mayer problem with fixed end-points by Gillie A. Larew‡ again leading to the analogs of the Legendre and Jacobi conditions. In a later paper§ the same author set up a definition of an invariant integral and a definition of Mayer fields for the Mayer problem. This made it possible to prove a theorem analogous to the Weierstrass theorem on

* See PALERMO RENDICONTI, vol. 29 (1910), p. 49.
† See TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), p. 459.
‡ See TRANSACTIONS OF THIS SOCIETY, vol. 20 (1918), p. 1.
§ Ibid., vol. 26 (1924), p. 61.

the total variation. Space is lacking for a discussion of the details of these papers.

## III. Multiple Integrals

1. *Parametric Form.* The study of the minima of multiple integrals has not reached the advanced stage which has been attained in the theory of the simple integral. The fundamental problem is concerned with an integral of the form

$$(27) \qquad \int_{(n)} f(x_1, \cdots, x_n, z, z_1, \cdots, z_n) dx_1 \cdots dx_n,$$

in which $z_i = \partial z/\partial x_i$, and in which the integral is to be extended over a closed manifold $M$ in the $n$-space determined by the coordinates $x_1, \cdots, x_n$. The function $f$ is of class $C'''$ in a domain $R_1$ of $(2n+1)$-space determined by the conditions that $(x_1, \cdots, x_n, z)$ be in a domain $R$ of $(n+1)$-space and that $z_i$ be finite. The question is then to determine among all functions $z$ of $x_1, \cdots, x_n$ which (1) are of class $C'$, (2) assume on the boundary of $M$ preassigned values, (3) yield points $(x_1, \cdots, x_n, z(x_1, \cdots, x_n))$ which lie in $R$, when $(x_1, \cdots, x_n)$ is in $M$, a function $Z$ for which there exists a positive number $d$, such that for any function $z$, which satisfies conditions (1), (2), (3), and also the condition that $|Z-z|<d$ for $(x_1, \cdots, x_n)$ in $M$, the integral (27) has a value not less (greater) than the value which it takes for $Z$.

By the same procedure which led to the Euler equation in the simple problem, one reaches the conclusion that the function $z$ must satisfy the partial differential equation

$$f_z - \sum \frac{\partial}{\partial x_i} f_{z_i} = 0.$$

A more symmetric form for this problem, and one which connects more readily with the theory of surfaces, is ob-

tained when parametric representation is introduced. We have then to consider the minimum of an integral

$$\int_{(n-1)} F(x_i, x_{i\mu}) du_1 \cdots du_{n-1}, \tag{28}$$

for $i=1, \cdots, n$, and $\mu=1, \cdots, n-1$, in which $x_{i\mu}=\partial x_i/\partial u_\mu$ and in which the integral is to be extended over a closed manifold in the $(n-1)$-space of the parameters $u_1, \cdots, u_{n-1}$. This integral has been studied for the case $n=3$, by Kobb;* and in the general case by Radon.† It is found that a minimizing surface must satisfy the partial differential equations

$$P_i \equiv F_{x_i} - \sum_1^{n-1} \frac{\partial}{\partial u_\mu} F_{x_{i\mu}} = 0,$$

which are not independent, but are connected by the $n-1$ relations $\sum_i P_i x_{i\mu}=0$.

The first question, which arises in the study of the problem in this form, refers to the conditions under which the integral (28) shall be independent of the choice of the parameters $u_1, \cdots, u_{n-1}$. The following treatment of this question is due to Radon. As in the case of the simple problem,‡ we find that if the integral (28) is to be invariant under a transformation of parameters $u_\mu=u_\mu(u_1', \cdots, u_{n-1}')$ which preserves the orientation upon a surface, i.e., such that

$$\Delta=\partial(u_1, \cdots, u_{n-1})/\partial(u_1', \cdots, u_{n-1}')>0,$$

then we must have

$$F(x_i, x_{i\mu}') = \Delta F(x_i, x_{i\mu}). \tag{29}$$

Radon observes that if $p_h$ represents the determinant obtained from the matrix $||x_{i\mu}||$ by omitting the $h$th row and prefixing the factor $(-1)^h$, then condition (29) is

* See Acta Mathematica, vol. 16 (1892), p. 65.

† See Monatshefte für Mathematik und Physik, vol. 22 (1911), p. 53.

‡ See I, §4(c).

equivalent to the condition that $F(x_i, x_{i\mu})$ can be expressed in terms of $x_i$ and $p_i$, $F(x_i, x_{i\mu}) = \Phi(x_i, p_i)$, and that for $k>0$, we shall have

$$\Phi(x_i, kp_i) = k\Phi(x_i, p_i);$$

a condition comparable in simplicity with the homogeneity condition (14) for the simple integral in parametric form. From this relation we obtain by differentiating with respect to $k$ and then putting $k=1$, the equation $\sum_i p_i \Phi_{p_i} = \Phi$, and hence by differentiating with respect to $p_j$ we find the system of equations $\sum_i p_i \Phi_{p_i p_j} = 0$. From this we conclude that there must exist $n(n-1)/2$ functions $\Phi_{rs}$ such that

$$\Phi_{p_i p_j} = \sum_{rs} x_{ir} x_{js} \Phi_{rs}, \qquad \Phi_{rs} = \Phi_{sr};$$

these functions $\Phi_{rs}$ play a role analogous to that of the function $F_1$ in the simple problem. Instead of to a set of $n$ partial differential equations between which there exist $n-1$ linear relations, the first order condition now leads to a single equation which a solution of the problem must satisfy, viz.,

$$T \equiv \sum_{i=1}^{n} \Phi_{x_i p_i} + \sum_{r,s=1}^{n-1} \sum_{h=1}^{n} \Phi_{rs} x_{hs} p_{hr} = 0,$$

an equation which resembles in form the Weierstrass equation for the simple problem

$$F_{xy'} - F_{x'y} + F_1(x'y'' - x''y') = 0.$$

Radon also obtains a simple expression for the transversality condition, a generalization of the Hilbert invariant integral, and of the Weierstrass formula.

The invariance of the integral (28) under parameter transformation has also been studied by Vivanti,* by Usai,†

* See PALERMO RENDICONTI, vol. 33 (1912), p. 268; ANNALI DI MATEMATICA, vol. 20 (1913), p. 49; PALERMO RENDICONTI, vol. 47 (1922), p. 232.

† See GIORNALE DI MATEMATICHE, vol. 52 (1914), p. 63; vol. 53 (1915), p. 136; LOMBARDO RENDICONTI, vol. 48 (1915), p. 77; ibid., vols. 49, 52; ANNALI DI MATEMATICA, vol. 31 (1922), p. 279.

and by Grosz.* Vivanti's earlier papers followed the method of Radon, giving, however, details of the reasoning which led to the conclusion that $F(x_i, x_{i\mu})$ must be expressible in the form $\Phi(x_i, p_i)$. Usai had attacked the problem by means of a partial differential equation for the function $F$ and had considered also the case in which $F$ contained partial derivatives of the $x_i$ of the second, third, and fourth orders. This was highly complicated work involving very elaborate calculations. In his 1922 paper, Vivanti developed a very simple method leading to a condition for the invariance of the simple integral $\int F(x, y, x', y', x'', y'', \cdots, x^{(n)}, y^{(n)})dt$, which may be stated as follows.

Let $\psi_1(t) = y'/x'$, and let $\psi_r(t)$ be defined by the recursion formula $\psi_r(t) = x'\psi_{r-1}(t) - (2r-3)x''\psi_{r-1}(t)$; these functions $\psi_r$ are the numerators of the formulas that express $d^r y/dx^r$ in terms of the derivatives with respect to $t$ of $x$ and $y$. Then the necessary and sufficient condition for the invariance of the integral is that $F$ be a positively (or negatively) homogeneous function of degree 1 with respect to $x', y', [\psi_2(t)]^{1/3}, \cdots, [\psi_r(t)]^{1/(2r-1)}, \cdots, [\psi_n(t)]^{1/(2n-1)}$. After the appearance of the paper, Usai, who had obtained an equivalent result by his elaborate methods, returned to the problem of the invariance under parameter transformation of an $n$-fold integral involving partial derivatives of higher order, viz.,

$$\int_{(n-1)} F(x_i, x_{i\mu_1}, x_{i\mu_1\mu_2}, \cdots, x_{i\mu_1\cdots\mu_k})du_\mu ,$$

for $\mu, \mu_1, \cdots, \mu_k = 1, \cdots, n-1$; and obtained a result of the same character as that of Vivanti. If $x_n$ be looked upon as a function of the remaining $n-1$ variables $x_1, \cdots, x_{n-1}$, we find that $\partial x_n/\partial x_\mu = -p_\mu/p_n$, where $p_\mu$ and $p_n$ are the functions defined by Radon. Similarly, it is found that

* See MONATSHEFTE FÜR MATHEMATIK UND PHYSIK, vol. 27 (1916), p. 70.

$$\frac{\partial^2 x_n}{\partial x_{\mu_1} dx_{\mu_2}} = \frac{\sigma_{\mu_1\mu_2}}{p_n{}^3}, \quad \cdots, \quad \frac{\partial^k x_n}{\partial x_{\mu_1} \cdots \partial x_{\mu_k}} = \frac{\sigma_{\mu_1\mu_2\cdots\mu_k}}{p_n^{2k-1}},$$

where $\sigma_{\mu_1\mu_2}, \cdots, \sigma_{\mu_1\mu_2\ldots\mu_k}$ are polynomials in the partial derivatives of $x_i$ up to those of order $2, \cdots, k$ respectively. The condition for invariance is now that $F$ be expressible as a function of $x_i, p_i, (\sigma_{\mu_1\mu_2})^{1/3}, \cdots, (\sigma_{\mu_1\mu_2\ldots\mu_k})^{1/(2k-1)}$ and that it be positively (or negatively) homogeneous of degree 1 with respect to $p_i, (\sigma_{\mu_1\mu_2})^{1/3}, \cdots, (\sigma_{\mu_1\mu_2\ldots\mu_k})^{1/(2k-1)}$.

These results answered definitively the question of the invariance of an $(n-1)$-fold integral involving $n$ functions of $n-1$ parameters. The question still remained open with regard to an $m$-fold integral, when $1<m<n-1$. This was answered, at least for integrals involving first partial derivatives, by Grosz in the paper referred to above. He considers the integral

$$\int_{(n)} F(z_k, z_{ki}) du_1 \cdots du_n,$$

for $k=1, \cdots, N=n+m$, and $i=1, \cdots, n$. Putting also $p_0=\partial(z_1, \cdots, z_n)/\partial(u_1, \cdots, u_n)$, and defining the symbol $p_{ij}(i=1, \cdots, n;\ j=1, \cdots, N-n)$ as the determinant obtained from $p_0$ by omitting $z_i$ and placing $z_{n+j}$ behind $z_n$, he finds that $F$ must be positively homogeneous of degree 1 with respect to $p_0$ and $p_{ij}$ at any point at which the matrix of the $z_{ki}$ is of rank $n$.

2. *Euler-Lagrange Rule; Jacobi's Condition.* The last mentioned paper offers a convenient transition to other parts of the theory of the multiple integral.

(a) The paper by Grosz is concerned primarily with an extension of the Lagrange problem to the double integral. The problems considered are variations on the following central theme of the paper.* To minimize the double integral $\int_B\int f(z_k; z_{kr})\, du_1 du_2$, $k=1, \cdots, n+2$; $r=1, 2$, when $z_k$ are to be functions of class $C''$ which satisfy the finite

* See the second footnote on p. 491.

equations $g_\pi(z_1, \cdots, z_{n+2})=0$, $\pi=1, \cdots, p<n$, and the isoperimetric conditions $\int_B\int f_\gamma(z_k; z_{kr})du_1du_2=C_\gamma$, $\gamma=1, \cdots, q$, and which take given values on the boundary of the domain $B$ over which the integration takes place. To insure non-singularity the requirement is made that the matrix $\|z_{kr}\|$ is everywhere of rank 2. For this problem, an extension of the multiplier rule is obtained by the methods developed by Hilbert and Bolza* for the classical Lagrange problem. The result may be stated as follows. If $z_k=Z_k(u_1,u_2)$ is a solution of the problem, then there must exist $q+1$ constants $l_0, \cdots, l_q$ and $p$ functions $\lambda_\pi(u_1,u_2)$ such that $Z_k$ and $\lambda_\pi$ satisfy the partial differential equations

$$F_{z_k}-\frac{\partial}{\partial u_1}F_{z_{k_1}}-\frac{\partial}{\partial u_2}F_{z_{k_2}}=0,$$

where $F\equiv l_0f+\sum l_\gamma f_\gamma+\sum\lambda_\pi g_\pi$.

The variations on this theme are obtained by modifications of the conditions which the functions $z_k$ have to satisfy on the boundary of the domain of integration $B$, and by the consideration of various possibilities for this boundary, of which only rectifiability is presupposed.

In spite of the considerable generality that is thus attained with respect to some elements of the problem, we are still far from having a theory of the general Lagrange problem for double integrals. Grosz goes further however and obtains an extension of the multiplier rule for the problem in which the integral $\int_B\int f(x,y,z,u,z_x,z_y,u_x,u_y)dxdy$ is to be minimized by functions which must also satisfy the partial differential equation $z_x=g(x, y, z, u, u_x)$.

(b) An interesting group of investigations is concerned with the analog of Jacobi's condition in the theory of the integral $\int_B\int f(x, y, z, p, q)dxdy$, where $p\equiv\partial z/\partial x$ and $q\equiv\partial z/\partial y$. As usually stated in the literature, this condition is that for no simple closed analytic curve $C_1$ which lies

* See the first footnote on p. 491.

entirely within $B$, shall there exist a function $\zeta(x,y)$, except $\zeta \equiv 0$, which satisfies the equation

$$\frac{\partial}{\partial x}(f_{pp}\zeta_x + f_{pq}\zeta_y) + \frac{\partial}{\partial y}(f_{qp}\zeta_x + f_{qq}\zeta_y) + \zeta\left(\frac{\partial}{\partial x}f_{xp} + \frac{\partial}{\partial y}f_{xq} - f_{xx}\right) = 0,$$

which is continuous on $C_1$ and of class $C''$ in the interior of $C_1$, and which vanishes on $C_1$. Because this condition is not readily applicable, it has been a desideratum to replace this condition by another in more usable form. One way in which this has been done recently* connects up most readily with the Jacobi condition III for the simplest problem. Writing $f_{y'y'} \equiv R$ and $f_{yy} - df_{yy'}/dx \equiv -A$, the Jacobi equation (8) takes the form $(Ru')' + Au = 0$. Consider now the boundary problem consisting of the differential equation

$$(Ru')' + \lambda Au = 0, \tag{30}$$

and the conditions $u(x_1) = 0$, $u'(x_1) = 1$. For this problem there exists an infinite set of positive characteristic constants $\{\lambda_n,\ n = 0,\ 1,\ \cdots\}$ and a corresponding system of functions $\{u_n(x)\}$, which satisfy the conditions of the problem and the further condition $u_n(x_2) = 0$. It has been shown by Picone† that a necessary and sufficient condition for the existence of a solution of the problem which vanishes exactly $n$ times on $(x_1, x_2)$ is that $\lambda_{n-1} \leqq \lambda < \lambda_n$, it being understood that we write 0 in place of $\lambda_{-1}$. Hence if $\lambda_0 > 1$, there exists no solution of Jacobi's equation which vanishes between $x_1$ and $x_2$; and if $\lambda_0 \leqq 1$, there exists such a solution of the equation. This result makes it possible to replace Jacobi's condition by the condition that the least positive

* See Picone, RENDICONTI DEI LINCEI, vol. 30 (1921), p. 410; also vol. 31 (1922), p. 46 and p. 94.

† See ANNALI SCUOLA NORMALE DI PISA, vol. 11 (1909), p. 3.

characteristic number $\lambda_0$ for the boundary problem (30) be greater than 1.

The proof of the necessity of the condition $\lambda_0 \geqq 1$ for a permanent sign of the second variation is simply made. For, $\lambda_0$ being less than 1, let us suppose that $\lambda_{\nu-1} < 1 \leqq \lambda_\nu$. Multiplying the equation $(Ru_n')' + \lambda_n A u_n = 0$ through by $u_n$, and integrating, we find

$$-\lambda_n \int_{x_1}^{x_2} A u_n^2 \, dx = \int_{x_1}^{x_2} u_n (R u_n')' dx$$

$$= u_n u_n' R \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} R u_n'^2 dx,$$

or

$$\int_{x_1}^{x_2} R u_n'^2 dx = \lambda_n \int_{x_1}^{x_2} A u_n^2 \, dx.$$

Hence, if Legendre's condition for a minimum is satisfied, $\int_{x_1}^{x_2} A u_n^2 dx > 0$; and moreover, we can show that for $n < \nu$, $\int_{x_1}^{x_2} (R u_n'^2 - A u_n^2) dx < 0$, while $\int_{x_1}^{x_2} (R u_n'^2 - A u_n^2) dx \geqq 0$ for $n \geqq \nu$. From this we conclude that if Jacobi's condition is not satisfied, the second variation can be made negative as well as positive by functions $u$ which vanish at $x_1$ and $x_2$.

The boundary problem (30) is equivalent to the linear integral equation*

$$u(x) = \lambda \int_{x_1}^{x_2} G(x, \xi) A(\xi) u(\xi) d\xi,$$

in which $G(x, \xi)$ is the corresponding Green's function; and the characteristic values $\lambda_n$ are the roots of the Fredholm determinant for this equation, $F(\lambda)$. We are therefore able to replace Jacobi's condition by the condition that $F(\lambda) \neq 0$ on (0, 1). It is this last form of Jacobi's condition which Picone has carried over to the theory of the double integral.

* See, e.g., Bôcher, *Leçons sur les Méthodes de Sturm*, p. 108 et seq.

The results which have here been indicated, and which run very close to investigations of Richardson,* had previously been given by Lichtenstein in papers† which carry the problem a great deal farther. Starting from the work of Schwarz on minimal surfaces,‡ he follows, for the general double integral, the method suggested above, and discusses also the case in which the least positive characteristic constant $\lambda_0$ is equal to 1. He proved also that if $\lambda_0 > 1$, the extremal can be imbedded in a field, and then extends the results to include the case in which the boundary of the minimizing surface is not fixed.

This latter case forms the subject of a recent doctoral dissertation by Simmons.§ He secured a very simple form for the second variation of the integral and deduced from this a formulation of the Jacobi condition in terms of a boundary value problem.

(c) Mention should finally be made of another method of dealing with the second variation of the double integral in parametric form, the object of which is to make a reduction of the second variation for this case to the form which it has in the non-parametric case. This reduction is based on a treatment of linear partial differential forms, similar to that used by the author for ordinary linear differential forms in the study of the second variation of the simple integral in parametric form.‖

3. *A General Formulation.* The discussion of the problems which have been mentioned thus far makes it possible to

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* See MATHEMATISCHE ANNALEN, vol. 68 (1910), p. 279; vol. 71 (1911), p. 214; TRANSACTIONS OF THIS SOCIETY, vol. 13 (1912), p. 22; vol. 18 (1917), p. 489.

† See SITZUNGSBERICHTE DER BERLINER GESELLSCHAFT, vol. 14 (1915), p. 119; MONATSHEFTE FÜR MATHEMATIK UND PHYSIK, vol. 28 (1917), p. 1; MATHEMATISCHE ZEITSCHRIFT, vol. 5 (1919), p. 26.

‡ See GESAMMELTE ABHANDLUNGEN, vol. 1, p. 223.

§ See Report of the April meeting of this Society, this BULLETIN, vol. 32 (1926), p. 221.

‖ See ANNALS OF MATHEMATICS, (2), vol. 13 (1912), p. 149; vol. 15 (1913), p. 78.

formulate now in general terms the problems of the calculus of variations involving integrals of various kinds. In every such problem we are concerned with an integral of the following form:

$$\int_M F\left(t_1, \cdots, t_m ; x_1, \cdots, x_n ; \frac{\partial x_\nu}{\partial t_\mu} ; \cdots ; \frac{\partial^r x_\nu}{\Pi_{\rho=1}^{r} dt_{\mu_\rho}}\right) dt_1 \cdots dt_m,$$

in which $m \leqq n$, and $r$ is an arbitrary integer, and in which the function $F$ is subject to conditions of continuity. The integration is extended over a closed domain $M$ in the $m$-dimensional space of the variables $t_\mu$, which may be completely determined, or which may merely be conditioned by restrictions that determine it incompletely.

In the second place we are concerned with a class $K_0$ of sets of functions $x_1(t_\mu), \cdots, x_n(t_\mu)$, or with a class $\overline{K}_0$ of $m$-dimensional manifolds in $n$-dimensional space $(m<n)$, represented by a set of functions $x_\nu(t_\mu)$ or by a set of functions $X_\nu(\tau_\mu)$, where

$$X_\nu(\tau_\mu) = x_\nu[T_1(\tau_1, \cdots, \tau_m), T_2(\tau_1, \cdots, \tau_m), \cdots, T_m(\tau_1, \cdots, \tau_m)],$$

in which the functions $T_\mu(\tau_1, \cdots, \tau_m)$ are of class $C'$ and such that $\partial(T_1, \cdots, T_m)/\partial(\tau_1, \cdots, \tau_m) > 0$. The class $K_0$ is further restricted by conditions on the functions which make up the set $x_1, \cdots, x_n$; these conditions may be in the first place continuity conditions, and further they may take the form of finite equations, or of differential equations, ordinary or partial; finally they include conditions which these functions must satisfy on the boundary of the closed domain $M$ over which the integration is extended.

Finally we are concerned with a subclass $K_1$ of the class $K_0$; this subclass usually is defined by means of such sets of functions $x_1, \cdots, x_n$ as lie within a prescribed neighborhood of a particular one of these sets. This neighborhood may be

of order 0, 1, $\cdots$, $p$, according as the conditions defining it include 1, 2, $\cdots$, $p+1$ of the inequalities

$$|y_\mu - x_\mu| < \rho, \quad |y_\mu' - x_\mu'| < \rho, \quad \cdots, \quad |y_\mu^{(p)} - x_\mu^{(p)}| < \rho.$$

With these elements in mind, we may now state the general problem to be one in which we ask for an element of class $K_0$, which shall give to the integral a value not greater (less) than that which it has for any element of class $K_1$. If $K_1 \equiv K_0$, we have the problem of an absolute minimum (maximum); if $K_1$ is a proper subclass of $K_0$ we have a problem of a relative minimum (maximum), the order of the problem being determined by the type of neighborhood involved in the definition of $K_1$.

## IV. Absolute Minima and Functional Calculus

1. *Minima of Functionals.* In 1904, Hilbert formulated the following general minimum problem. An infinite class $K$ of mathematical objects $a$, $b$, $\cdots$ is given; also an operator $J$ associating with each element a real number. To determine an element of $K$ to which corresponds the smallest number.* Apart from the evident vagueness of this problem, it is clear that it includes a vast number of problems many of which are not, as far as we can see now, problems of the calculus of variations and are not reducible to such problems. Among them are the ordinary maxima and minima problems, problems like that of Kakeya† which asks for the least area in which it is possible to turn a line of given length through an angle of 180° and also the type of problem which has been discussed by Bonnesen,‡ Lebesgue,§ Blaschke‖ and others. It includes the general problem of the minima and maxima of functionals. And the

* See Bolza, *Vorlesungen*, p. 16, footnote.

† See, e.g., Ford, this Bulletin, vol. 28 (1922), p. 45.

‡ See Mathematische Annalen, vol. 91 (1924), p. 252; vol. 95 (1925), p. 267.

§ See Journal de Mathématiques, (8), vol. 4 (1921), p. 67.

‖ See Mathematische Zeitschrift, vol. 6 (1920), p. 281.

calculus of variations may indeed be conceived as a part of the broader theory.

This standpoint, clearly suggested by the work of Volterra, was definitely introduced in the calculus of variations by Hadamard,* but only in a tentative way. It has been systematically developed by Tonelli, in his *Fundamenti di Calcolo delle Variazioni*, to which the greater part of the present section will be devoted. In the discussion of the Euler-Lagrange multiplier rule we have already called attention to the extension of this rule by Hahn† to the case in which the integrals are replaced by more general functionals. Mention must also be made of a paper by LeStourgeon,‡ which is concerned with minima of functionals (in particular of functions of lines), of such character as to include the integrals of the calculus of variations. The independent variables are arcs of curve defined by equations

$$y = \lambda(x), \qquad a \leqq x \leqq b.$$

For each arc $\lambda$ of a class $L$ of such arcs there is defined a real number $F(\lambda)$. The problem considered is that of determining an arc $\lambda_0$ of $L$ for which there shall exist a number $d$, of such character that for every arc of $L$ for which $|\lambda-\lambda_0| < d$, $|\lambda'-\lambda_0'| < d$, we shall have $F(\lambda) \geqq F(\lambda_0)$. We say that $F(\lambda)$ has a differential at $\lambda_0$ if there exists a linear functional $L(\eta)$ with continuity of order 1, such that

$$F(\lambda_0+\eta) - F(\lambda_0) = L(\eta) + \epsilon(\eta) M_1(\eta),$$

where $M_1(\eta)$ is the maximum $|\eta|$ and $|\eta'|$ on $(a, b)$, and $\epsilon(\eta) \to 0$ with $M_1(\eta)$. Similarly, $F(\lambda)$ is said to have a second differential at $\lambda_0$ if there exists a linear functional $L(\lambda)$ and a bilinear functional $B(\lambda, \mu)$, such that

$$F(\lambda_0 + \eta) - F(\lambda_0) = L(\eta) + B(\eta, \eta) + M^2(\eta) \cdot \epsilon(\eta),$$

where $\epsilon(\eta)$ and $M(\eta)$ are defined as before.

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* See *Leçons sur le Calcul des Variations*, Book 2, Chap. 7.

† See II, §2(c).

‡ See TRANSACTIONS OF THIS SOCIETY, vol. 21 (1920), p. 357.

By a modification of the theory of Riesz, referred to above,* it is shown that a linear functional $L(\lambda)$ which has continuity of order 1, is expressible, in an infinitude of ways, in the form

$$L(\lambda) = \int_a^b \lambda(x)du(x) + \int_a^b \lambda'(x)du_1(x),$$

where $u(x)$ and $u_1(x)$ are functions of limited variation on $(a, b)$; and also that a bilinear functional $B(\lambda, \mu)$ which has continuity of order 1 in each variable, when the other is fixed, can be expressed in the form

$$B(\lambda,\mu) = \int_a^b\int_c^d \lambda(x)\mu(y)d_{xy}p(x,y) + \int_a^b\int_c^d \lambda'(x)\mu(y)d_{xy}q'(x,y)$$
$$+ \int_a^b\int_c^d \lambda\mu' d_{xy}q''(x,y) + \int_a^b\int_c^d \lambda'\mu' d_{xy}r(x,y)$$

in which $p$, $q'$, $q''$, $r$ are of limited variation in $x$ and $y$ separately and jointly.

It is now shown that conditions of the first and second orders for a minimum of $F(\lambda)$, which is supposed to possess a second differential, are obtained from the conditions $L(\eta)=0$ and $B(\eta, \eta)\geqq 0$; and from these, generalizations of the Euler equation, of the transversality condition and of the Jacobi condition are obtained, the latter by applying Bliss' method† to the condition $B(\eta, \eta)\geqq 0$.

2. *Introduction to Tonelli's Work.* In the two volumes of Tonelli's *Fondamenti* which have thus far appeared, the simplest problem of the calculus of variations is discussed, both in the form (1) and in the parametric form (13), with and without an isoperimetric condition. We shall confine our brief account of this work to the theory of the unrestricted integral (13). This integral associates a real number with every curve $C$ of a certain class, and is therefore a function of such curves; the integral is studied with regard to its dependence upon the curve.

* See the footnote on p. 496.

† See I, §5 (b).

The various problems of the calculus of variations which have been discussed in earlier sections, are concerned with the relative extremes of an integral; i.e., they are problems for which the class $K_1$ is a proper subclass of $K_0$. To secure knowledge of the absolute extremes, one might conceivably, after having determined all relative extremes, determine that one among them, which gives to the integral the smallest value.

But this method leaves out of account the difficulty that arises in case there is an infinitude of relative extremands, in which case one cannot be sure that any absolute extreme exists. The question as to the existence of an absolute extreme has given rise to the famous Dirichlet principle, according to which it was argued that

$$\iint (u_x^2 + u_y^2)\, dxdy$$

must have a minimum, because its value is always $\geqq 0$. That this conclusion involves a confusion between minimum and greatest lower bound was pointed out by Weierstrass.

A first step in the direction of establishing conditions which will insure the existence of an absolute extreme was taken by Hilbert,* who proved an existence theorem, which has since been extended to the following form.

If (1) $F(x, y, x', y')$ is of class $C'''$ and positively homogeneous of degree 1 in $x'$ and $y'$ in a region $T$, defined by $(x, y)$ in $R$, $x'^2+y'^2 \neq 0$,

(2) $F(x, y \cos \gamma, \sin \gamma) > 0$ for $(x, y)$ in $R_0$, a domain in the interior of $R$, and $\gamma$ arbitrary,

(3) $F_1(x, y, \cos \gamma, \sin \gamma) > 0$, for $(x, y)$ in $R_0$ and $\gamma$ arbitrary,

(4) $R_0$ is bounded, closed, and convex,

(5) $A_1$ and $A_2$ are two distinct points of $R_0$.

* See JAHRESBERICHT DER VEREINIGUNG, vol. 8 (1899), p. 184; see also Bolza, loc. cit., Chap. IX.

Then there exists at least one rectifiable curve $H$ joining $A_1$ and $A_2$, which furnishes an absolute minimum for a suitably generalized integral

$$\int F(x, y, x', y')dt,$$

with respect to the totality of rectifiable curves which join $A_1$ and $A_2$ and which lie in $R_0$. A large part of Tonelli's second volume is devoted to existence theorems of this general character. That we have a problem here very different from the corresponding one in the theory of functions of a real variable becomes clear when we realize that the functionals of the calculus of variations do not possess the continuity property which is basal in the other theory. A very wide class of integrals of the calculus of variations possess lower semicontinuity, which property accordingly occupies a central position in Tonelli's treatment.* In order to obtain an idea of the scope of this treatment, we must take up a few agreements and definitions. The integral (13) taken along a curve $C$ is denoted by $I_C$.

(1) A point set $A$ of the $xy$-plane, such that those of its points which belong to any circle form a *closed* set, is called a region. The function $F(x, y, x', y')$ is understood to be of class $C$, and of class $C''$ with respect to $x'$ and $y'$ for $(x, y)$ in $A$ and $x'^2+y'^2\neq 0$; and also to satisfy the homogeneity condition (14).

(2) If $F_1>0(<0)$ for $(x, y)$ in $A$ and $x'^2 + y'^2\neq 0$, the integral $I_C$ is called *positively (negatively) regular*; if $F_1\geqq 0(\leqq 0)$, $I_C$ is called *positively (negatively) quasi-regular.*

(3) If $F_1\geqq 0$ for $(x, y)$ in $A$ and $x'^2+y'^2\neq 0$, and if for *no point* $(x, y)$ *in* $A$, the values of $\theta$ for which $F_1(x, y, \cos\theta, \sin\theta)=0$ fill up any subinterval of $(0, 2\pi)$, $I_C$ is called *normally quasi-regular.*

(4) If $F_1\geqq 0$ for $(x, y)$ in $A$ and $x'^2+y'^2\neq 0$, and if for *no point* in $A$, we have $F_1(x, y, \cos\theta, \sin\theta)=0$ for all $\theta$, $I_C$ is called *semi-normally quasi-regular.*

* See an article by Tonelli, this BULLETIN, vol. 31 (1925), p. 163.

(5) If $F>0(<0)$ for $(x, y)$ in $A$ and $x'^2+y'^2\neq 0$, $I_C$ is called *positively (negatively) definite*; if $F\geqq 0(\leqq 0)$, $I_C$ is called *positively (negatively) semi-definite.*

(6) An *ordinary* curve is one which lies in the region $A$ and which is rectifiable.

(7) A class $K$ of ordinary curves is called *complete* if every *rectifiable accumulation curve* of the set also belongs to the set.

3. *Tonelli's Treatment.* With these definitions in mind, we proceed to the following theorem.

(a) If $I_C$ is positively quasi-regular, and if for a given complete class $K$ of ordinary curves $C$, all in a bounded portion $A_1$ of $A$, it is possible to determine a function $\Phi(\alpha)$, defined and continuous for all real values of $\alpha$, always non-negative and non-decreasing, such that for any curve $C$ of $K$, we have

$$L\leqq\Phi(I_C),$$

in which $L$ is the length of $C$, then $I_C$ has an absolute minimum in $K$.

We shall indicate the principal steps in the proof of this theorem, so as to obtain an insight into Tonelli's method and because it acquaints us with some of the central features of his theory. Let $\{C_n\}$, $n=1, 2, \cdots$, be a sequence of sets of ordinary curves taken from $K$, such that for every curve $C_n$ of the set $\{C_n\}$, we shall have

$$I_{C_n}<i+\frac{1}{n}, \quad \text{or} \quad I_{C_n}<-n,$$

$i$ being the greatest lower bound of the values of $I_C$ in $K$, according as $i$ is finite or infinite, this greatest lower bound being necessarily equal to $-\infty$ in the latter case. We show then first of all that $i$ is finite; for in the contrary case we would have in virtue of the hypotheses on $\Phi(\alpha)$, if $L_n$ denotes the length of $C_n$,

$$L_n\leqq\Phi(I_{C_n})\leqq\Phi(-n)\leqq\Phi(-1).$$

Hence, if $M$ is the maximum of $|F(x, y, \cos\theta, \sin\theta)|$ for $(x, y)$ in $A_1$ and for $0 \leqq \theta \leqq 2\pi$, we shall have

$$I_{C_n} > -ML_n \geqq -M\Phi(-1);$$

but this result contradicts the condition $I_{C_n} < -n$.

We conclude that we have for every curve $C_n$ of the set $\{C_n\}$, $I_{C_n} < i + 1/n$; and consequently

$$L_n \leqq \Phi(I_{C_n}) \leqq \Phi\left(i + \frac{1}{n}\right) \leqq \Phi(i+1),$$

so that the lengths of the curves of the sets $\{C_n\}$ are bounded. Now we make use of an important theorem of Hilbert* according to which an "infinite set of continuous curves, all contained in a bounded region and whose lengths form a bounded set, has at least one continuous and rectifiable accumulation curve." This theorem has played an important part in Hilbert's existence proofs, and it is not surprising to see it appear here. We conclude from it that the set of curves $K$ has a rectifiable accumulation curve $C_0$. It is readily proved that $C_0$ must lie within the region $A$ and that its length $L_0$ is less than or equal to $\Phi(i+1)$; hence, since the class $K$ is complete, this curve $C_0$ belongs to $K$.

Now Tonelli has proved† that "If $I_C$ is positively quasi-regular, and $L$ is an arbitrary positive number, then $I_C$ is a lower semi-continuous function on the set of ordinary curves whose length is less than $L$." From this it follows that $I_C$ is lower semi-continuous on $C_0$, which means that for any $\epsilon$, there exists a $\rho$, such that for any ordinary curve $C$ which lies in the $\rho$-neighborhood of $C_0$, we have

$$I_{C_0} < I_C + \epsilon.$$

In particular, therefore, there exists an $n_\epsilon$, such that if $n > n_\epsilon$, there is at least one curve $C_n$ for which

$$I_{C_0} < I_{C_n} + \epsilon.$$

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* See JAHRESBERICHT DER VEREINIGUNG, vol. 8 (1900), p. 184.

† See *Fondamenti*, vol. I, p. 292.

Consequently, $I_{C_0} < i + 1/n + \epsilon$, for every $n > n_\epsilon$; hence we conclude $I_{C_0} \leqq i + \epsilon$, and therefore $I_{C_0} \leqq i$. On the other hand, since $C_0$ is in $K$, we must have $I_{C_0} \geqq i$; therefore we obtain the final result that $I_{C_0} = i$, which means that the rectifiable curve $C_0$ furnishes an absolute minimum for $I_C$.

(b) There is a second criterion for the existence of an absolute minimum of $I_C$, for the case that $I_C$ is positively quasi-regular and positively semi-definite; the proof depends in an essential manner on the fact that also in this case $I_C$ is lower semi-continuous. From these two criteria, twelve existence theorems for an absolute minimum are derived which form a nucleus for the further developments and which constitute therefore a very fundamental part of the work. Thus it is seen how essential a role the lower semi-continuity of $I_C$ plays in Tonelli's theory of the absolute minimum, and it is not surprising therefore that the first volume of his work has as its central purpose the study of this property. There are given a large number of necessary conditions, of sufficient conditions, and of conditions which are both necessary and sufficient for the lower semi-continuity of $I_C$, over the entire class of ordinary curves (i. e. uniform lower semi-continuity), over a restricted class of curves, or on a particular curve. Among the results that are obtained it is interesting to note the following:

(1) If $I_C$ is positively quasi-regular and positively definite, it is lower semi-continuous.

(2) If $I_C$ is lower semi-continuous in $A$, then $F_1(x, y, \cos\gamma, \sin\gamma) \geqq 0$ for every $\gamma$ and for every point $(x, y)$ which is interior to $A$ or is a limiting point of interior points of $A$.

(3) If $A$ satisfies a certain condition of convexity, and if $I_C$ is positively definite, then a necessary and sufficient condition for the lower semi-continuity of $I_C$ on an ordinary curve $C_0$ in $A$ is that for "nearly every point" of $C_0$ and for all values of $\theta$, we have

$$E(x_0, y_0, \cos\theta_0, \sin\theta_0, \cos\theta, \sin\theta) \geqq 0 .$$

4. *Further Developments.* Once the question as to the existence of a curve which furnishes an absolute minimum has been answered more or less completely, attention is turned to the properties of such curves.

(a) From the conditions for semi-continuity and the relation of this property to the existence of minima, the necessity of the conditions of Legendre and Weierstrass follows without difficulty. It is next shown in the usual way that at every point of a minimizing curve of class $C'$ which lies in the interior of the region $A$, the Euler equations must be satisfied; also, that every ordinary extremand which is not of class $C'$ must satisfy the equations

$$\int_0^s F_x\,ds - \frac{d}{ds}\int_0^s F_{x'}\,ds = C_1, \qquad \int_0^s F_y\,ds - \frac{d}{ds}\int_0^s F_{y'}\,ds = C_2,$$

along every one of its arcs whose points fall entirely within $A$; a curve satisfying these equations is called an extremaloid.

(b) When we come to the question of the existence of extremals and extremaloids, it is clear that this can be answered by means of the existence theorems for absolute minima. For a variety of conditions are known under which there must exist at least one minimizing curve; moreover, such a curve must be an extremal or an extremaloid—hence we can infer the existence of extremals (extremaloids).

We observe that this procedure suggests a new method for proving the existence of solutions of differential equations of the second order, since such an equation may always be looked upon as the Euler equation of some problem of the calculus of variations. At any rate the existence of extremals is put on a basis independent of the existence theorems for differential equations. In consequence of this the problems concerning the uniqueness of extremals, their dependence on the initial points and their differentialibity with respect to initial data receive a treatment which does not, as usually has been the case, depend upon the theory of differential equations. And finally, necessary and suffi-

cient conditions for relative minima are derivable from those for absolute minima; thus a connection is established with the classical theory.

(c) For the simplest problems of the calculus of variations, Tonelli's method dominated by the point of view of the functional calculus, has led to an important re-orientation of the subject, and to the establishment of an imposing group of existence theorems. How fruitful it will prove to be in the study of more complicated problems, and whether it will enable us to study functionals of more general character than those included in the calculus of variations, are questions which I am not able to answer. Certainly, our knowledge concerning the minima of multiple integrals, at present handicapped by the incompleteness of our knowledge concerning partial differential equations, would be greatly enlarged if the functional methods were to prove extensible so as also to cover these cases.

## V. Appendix

In this report on progress in the calculus of variations, I have of necessity limited myself to a few of the subdivisions of this field. No mention has been made, e. g., of the applications and extensions which are being made in the newer differential geometry of calculus of variations concepts. Moreover, I could not aspire to completeness in those parts of the subject with which I did deal. While very useful bibliographies for the calculus of variations up to 1920 have been published by Lecat,* it will be worth while to list here, although merely by title, some additional papers which have come to my attention, and which are not mentioned by Lecat, most of them being subsequent to 1920.

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* See *Bibliographie du Calcul des Variations, depuis les origines jusqu'à* 1850, Paris, Hermann; *Bibliographie du Calcul des Variations*, 1850–1913, Paris, Hermann; Appendix to the *Bibliographie des Séries Trigonométriques*, published by the author, Avenue des Alliés, 92, Louvain, which carries the bibliography of the calculus of variations up to 1920.

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# NOTE ON THE MERSENNE NUMBER $2^{139}-1$*

BY D. H. LEHMER

Lucas' test for the primality of the Mersenne number $2^n-1$, $n$ a prime number, has been applied to the number $2^{139}-1$. For $n=139$ we form the residues, modulo $2^{139}-1$, of the successive terms of the recurring series 3, 7, 47, 2207, $\cdots$, in which each terms is 2 less than the square of the preceding term. Then $2^{139}-1$ is composite if no one of the first 138 terms of this series is divisible by $2^{139}-1$. Since this proved true, this 42-digit number is composite; no factor of it is known although the search for such has been pushed to the limit 1,000,000 by Cunningham and Gérardin.†

Contrary to the suggestions of Lucas and other workers in this field, the test was made to the base ten instead of to the base two. Lucas' method,‡ in which the main operation is copying, would require the setting down of more than 2,500,000 digits. Mason§ has suggested an abbreviation of this method in which the copying is replaced by the quicker but more dangerous procedure of counting. Both these methods are too slow for dealing with numbers as large as $2^{139}-1$. The method employed in the present case was a variation of the writer's "cross-division"|| process adapted to meet the requirements of the calculation. The work was done entirely by a computing machine in 60 hours. Since Lucas' test for a composite Mersenne number is a negative one it is vital that no error be made. To minimize the chance of error, two checks were made at each step by casting out multiples of 1001.

It should be noted that the composite character of $2^{139}-1$ is in accord with the famous statement of Mersenne in 1644.

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* Presented to the Society, San Francisco Section, June 12, 1926.

† BRITISH ASSOCIATION REPORTS, 1911, p. 321.

‡ *Récréations Mathématiques*, vol. 2, 1883, p. 230.

§ PROCEEDINGS INDIANA ACADEMY OF SCIENCE, 1914, pp. 429–31.

|| AMERICAN MATHEMATICAL MONTHLY, 1926, p. 198.

# ON THE SERIAL RELATION IN BOOLEAN ALGEBRAS*

BY B. A. BERNSTEIN

The elements $x, y, z, \cdots$ of a class $K$ are said to form a *series* with respect to a dyadic relation $S$, if they satisfy the following postulates:†

$P_1$. If $x \neq y$, then either $xSy$ or $ySx$.

$P_2$. If $xSy$, then $x \neq y$.

$P_3$. If $xSy$ and $ySz$, then $xSz$.

It is my object to determine all serial relations in Boolean algebras given by universal propositions expressible in the fundamental Boolean operations of addition, multiplication, and negation.

All the desired serial relations $S$ must be of the form

$$axy + bxy' + cx'y + dx'y' = 0, \tag{1}$$

where $x'$ is the negative of $x$. Our problem then reduces itself to finding the conditions imposed on the coefficients of (1) by $P_1$-$P_3$. We proceed to determine these conditions.

If in (1) $x=0$ and $y=1$, then $c=0$; if $x=1$ and $y=0$, then $b=0$. Hence the condition imposed on (1) by $P_1$ is

$$b = 0 \quad \text{or} \quad c = 0. \tag{2}$$

The condition imposed on (1) by $P_2$ is that the equation $axx + bxx' + cx'x + dx'x' = 0$ have no solution. This condition is

$$ad \neq 0. \tag{3}$$

The condition that (1) satisfies $P_3$ non-vacuously is (this BULLETIN, vol. 30, p. 127)

$$a + d < b + c, \quad ad = 0, \tag{4}$$

*which contradicts* (3). Hence, *there are no non-vacuous serial relations* (1) *in any Boolean algebra.*

* Presented to the Society, San Francisco Section, June 12, 1926.

† See E. V. Huntington, *The Continuum*, 2d ed., p. 10.

But there may possibly be *vacuous* serial relations (1). To determine these, let us first find the condition that there be no $x, y, z$ such that both $xSy$ and $ySz$ hold, i. e., the condition that there be no $x, y, z$ such that simultaneously

(i) $$axy + bxy' + cx'y + dx'y' = 0,$$

(ii) $$ayz + byz' + cy'z + dy'z' = 0.$$

Since (i) and (ii) together are equivalent to the single equation

(iii) $$axyz + (b + c)xy'z + (b + c)x'yz + (c + d)x'y'z + (a+b)xyz' + (b+d)xy'z' + (b+c)x'yz' + dx'y'z' = 0,$$

our condition is found to be

(5) $$ad(b+c) \neq 0.$$

Now $a$ in (1) must be either 0 or 1, for otherwise neither $1Sa$ nor $aS1$, contrary to $P_1$. Similarly $b, c, d$ must be either 0 or 1. Hence, taking account of (2) and (5), we find that all vacuous serial relations (1) must be among the relations

(6) $$xy + xy' + x'y' = 0, \qquad xy + x'y + x'y' = 0.$$

Since for $e \neq 0, 1$ neither equation of (6) is satisfied by $x = 1$ and $y = e$, or by $x = e$ and $y = 1$, we see that *there are no vacuous serial relations* (1) *in a Boolean algebra consisting of more than two elements.*

However, equations (6) are serial relations in a *two-element* Boolean algebra, since they produce respectively the relation-tables (this BULLETIN vol. 30, p. 27):

(7)

| | 0 | 1 |
|---|---|---|
| 0 | − | + |
| 1 | − | − , |

| | 0 | 1 |
|---|---|---|
| 0 | − | − |
| 1 | + | − . |

We may sum up our results as follows: *Relations* (6) *are the only serial relations of form* (1) *in Boolean algebras; both of these relations are vacuous and both are confined to the Boolean algebra of order two.*

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# NOTE ON HOROSPHERES*

BY JAMES PIERPONT

In a paper† before the Mathematical Association of America (Sept. 1923) and in another before the International Mathematical Congress at Toronto (August, 1924) I have developed a treatment of a part of non-euclidean geometry which has certain advantages, so it seems to me, over the projective methods of Klein. This method rests on the metric of Riemann and on the introduction of certain variables in terms of which the equations of the straight line and plane are linear.

In the first paper mentioned I showed how easy it is to arrive at Clifford's parallels in elliptic space. These lie on surfaces called Clifford surfaces; their curvature is zero and hence their geometry for restricted regions is euclidean.

In hyperbolic space there are also surfaces of zero curvature, the horospheres of Lobatschevsky and Bolyai. I now wish to show how they may be obtained by the preceding method.

Let $x$, $y$, $z$ be ordinary rectangular coördinates. Let $R$ be a positive constant. We set

$$r^2 = x^2 + y^2 + z^2,\ \lambda = 4R^2 - r^2,\ \mu = 4R^2 + r^2,$$

$$d\sigma^2 = dx^2 + dy^2 + dz^2. \tag{1}$$

The metric of $H$-space‡ as defined by Riemann is given by the equation

$$ds = \frac{4R^2 d\sigma}{\lambda}. \tag{2}$$

* Presented to the Society, May 1, 1926.

† AMERICAN MATHEMATICAL MONTHLY, vol. 30, p. 425, and vol. 31, p. 26. The PROCEEDINGS OF THE TORONTO CONGRESS have not yet appeared.

‡ For $H$–, read hyperbolic; for $e$-, read euclidean.

Here $R$ plays the role of the *space constant* and $-1/R^2$ the curvature of this space. Since $ds<0$ for $e$-points without the $e$-sphere $\lambda=0$, we restrict ourselves in $H$-geometry to points within it. The distance from any point within $\lambda$ to a point on it is infinite. It is the boundary of $H$-space. Straight lines or $H$-straights are defined by

$$\delta\int ds = 0.$$

They turn out to be $e$-circles cutting the $\lambda$-sphere orthogonally.

In many problems it is convenient to introduce the variables

$$(3)\qquad z_1 = \frac{4R^2x}{\lambda},\quad z_2 = \frac{4R^2y}{\lambda},\quad z_3 = \frac{4R^2z}{\lambda},\quad z_4 = R\mu/\lambda.$$

In problems of the plane we may set $z=0$ in $r$, $\lambda$, $\mu$ in (1) and use

$$(4)\qquad z_1 = \frac{4R^2x}{\lambda},\quad z_2 = \frac{4R^2y}{\lambda},\quad z_3 = R\mu/\lambda.$$

The $\lambda$ sphere is then replaced by the $\lambda$-circle, $x^2+y^2-4R^2=0$. With this introduction, let us consider $H$-circles, in the $xy$ plane. Their equation is in $z$-coordinates

$$(5)\qquad a_1z_1 + a_2z_2 - a_3z_3 = k,\ \text{a constant},$$

the center having the $z$-coordinates $a_1$, $a_2$, $a_3$. In $xy$ coordinates (5) becomes

$$(6)\qquad x^2 + y^2 + \frac{4R^2}{k - a_3R}(a_1x + a_2y) - \frac{4R^2(k + a_3R)}{k - a_3R} = 0,$$

which is an $e$-circle.

We ask what happens when the center $a$ converges to a point on the $\lambda$-circle. To answer this question we consider the family of $e$-circles

$$(7)\qquad C_\alpha = x^2 + y^2 - 2\alpha x - 4R(R - \alpha) = 0.$$

whose centers lie on the $x$ axis and which touch the $\lambda$-circle at the point $A$; also the $e$-circles

$$L_\beta = x^2 + y^2 - 4Rx - 2\beta y + 4R^2 = 0. \tag{8}$$

These cut the $\lambda$-circle orthogonally and are therefore $H$-straights. They also pass through the point $A$ and cut the $C_\alpha$ orthogonally.

We show that the $C_\alpha$ may be regarded as $H$-circles whose center is $A$ and whose radii are the $L_\beta$. In fact let $L_\beta$ cut $C_\alpha$ and $C_{\alpha_1}$ in $P$, $P_1$; we show the $H$-length of the segment $P\,P_1$ on the radius $L_\beta$ is independent of $\beta$.

For if $L_\beta$ cuts $C_\alpha$ in the point $P$ whose coordinates are $x, y$ we find that

$$\xi = x - 2R = \frac{2\beta^2 c}{\gamma^2},\quad y = \frac{2\beta c^2}{\gamma^2}, \tag{9}$$

where

$$c = \alpha - 2R,\quad \gamma^2 = \beta^2 + c^2. \tag{10}$$

If $L_\beta$ cuts the adjacent circle $C_{\alpha+d\alpha}$ in $P'$, we find the length of the arc $P\,P'$ is

$$ds = \frac{2R^2 d\alpha}{\alpha(2R - \alpha)};$$

hence

$$\overline{PP_1} = \int ds, \tag{11}$$

which is independent of the particular radius $L_\beta$.

We may also show that the $H$-circles (5) or (6) converge to the circles $C_\alpha$. In fact take $a$ on the $x$ axis for simplicity; then $a_2 = 0$. We may now let $a_1, a_3, k \to \infty$ such that

$$\lim_{k\to\infty} \frac{a_1}{k} = \lim_{k\to\infty} \frac{a_3}{k} = \frac{1}{R} \cdot \frac{\alpha}{\alpha - 2R}. \tag{12}$$

In this case (6) goes over into a $C_\alpha$ circle. These $C_\alpha$ circles are horocircles. For simplicity we took their $H$-center $A$ on the $x$ axis; obviously $A$ may be any point on the

$\lambda$-circle. Any $e$-circle lying within $\lambda=0$ and tangent to it, is a *horocircle* in $H$-geometry.

To get a *horosphere* we merely revolve one of the horocircles about the diameter through $A$; i.e., an $e$-sphere lying within the $\lambda$-sphere or $x^2+y^2+z^2-4R^2=0$, and tangent to it is a horosphere. We wish to show that these surfaces have 0-curvature or that the metric on them has the form

$$ds^2 = \frac{4R^4c^2}{\alpha}(du^2 + u^2d\varphi^2). \tag{13}$$

To this end let us revolve a $C_\alpha$ about the $x$ axis getting a horosphere $S_\alpha$. The intersection $P$ of $C_\alpha$ and some $L_\beta$, describes an $e$-circle which we may regard as a parallel of latitude $\beta$ whose pole is $A$ and we may use this as one coordinate. As second coordinate on $S_\alpha$ we may use the meridian circles $\varphi$ cut out of $S_\alpha$ by $e$-planes through the $x$-axis and making the angle $\varphi$ with the $xz$ plane. If now we set $\rho^2=y^2+z^2$, we have, using (9), (10),

$$\rho = \frac{2\beta c^2}{\gamma^2}, \quad y = \rho\sin\varphi, \quad z = \rho\cos\varphi.$$

Now $ds$ is given by (2). We find

$$\beta d\beta = \gamma d\gamma, \qquad dx = d\xi, \qquad dy^2 + dz^2 = d\rho^2 + \rho^2 d\varphi^2,$$

$$d\rho = \frac{2c^2}{\gamma^4}(c^2 - \beta^2)d\beta,\ d\xi = \frac{4\beta c^2}{\gamma^4}d\beta,\ d\sigma^2 = \frac{4c^4}{\gamma^4}(d\beta^2 + \beta^2 d\varphi^2).$$

Hence

$$ds^2 = \frac{4R^4c^2}{\alpha^2}\left(\frac{d\beta^2}{\beta^4} + \frac{d\varphi^2}{\beta^2}\right).$$

Set now $\mu=1/\beta$; then $ds^2$ reduces to (13); q.e.d.

YALE UNIVERSITY

# SOME THEOREMS CONCERNING MEASURABLE FUNCTIONS*

BY L. M. GRAVES†

Theorems on the measurability of functions of measurable functions, e. g., in the form $F(x)=f[x, g(x)]$, have been given by Carathéodory and other writers.‡ Our Theorem I is an easy generalization of the one given by Carathéodory on page 665, with a slightly different method of proof. Here the function $f(x, y)$ is supposed to be defined for all values of $y$. Our Theorem II merely applies Theorem I to certain cases when the function $f(x, y)$ is *not* defined for all values of $y$. In these theorems the variables $x$ and $y$ may be multipartite. Theorems I and II are still valid if, throughout, *measurable* is replaced by *Borel measurable.*

In Theorem III, we consider a summable function $f(x, y)$ of two variables, and show by means of Theorem I that the function of $x$ alone

$$\int_a^x f(x, y)dy$$

is also summable, under a suitable convention.

*Notations.* In Theorems I and II we use the following abbreviated notations: The point $(x_1, \cdots, x_k)$ in $k$-dimensional space, we denote simply by $x$. The $x$-space as a whole is denoted by the German $\mathfrak{x}$. We do similarly for the $m$-dimensional space $\mathfrak{y}$. When we have to speak of the $(k+m)$-dimensional space $(\mathfrak{x}, \mathfrak{y})$, we may denote

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† National Research Fellow in Mathematics.

‡ See Carathéodory, *Vorlesungen über reelle Funktionen*, pp. 376, 377, 665; Hans Hahn, *Theorie der reellen Funktionen*, p. 556. Hobson, *Theory of Functions of a Real Variable*, 2d ed., vol. 1, p. 518.

it by $\mathfrak{W}$. Corresponding to a set $\mathfrak{W}^{(0)}$ of points of the space $\mathfrak{W}$ and a point $y$ of the space $\mathfrak{Y}$, we denote by $\mathfrak{X}^{(y)}$ the set of all points $x$ such that $(x, y)$ is in $\mathfrak{W}^{(0)}$. The sets $\mathfrak{Y}^{(x)}$ are defined similarly. A set of $m$ functions $g_1(x), \cdots, g_m(x)$, each single-real-valued on a set $\mathfrak{X}^{(0)}$ of the space $\mathfrak{X}$, will be denoted simply by $g(x)$, and called a function *on* $\mathfrak{X}^{(0)}$ *to* $\mathfrak{Y}$. This function is said to be measurable on $\mathfrak{X}^{(0)}$ if each component is measurable. We denote by $[y]_a$ the closed neighborhood of the point $y$ consisting of all those points $\bar{y}$ distant from $y$ by not more than $a$.

THEOREM I. *Let $\mathfrak{X}^{(0)}$ be a measurable set, and let $f(x, y)$ be a single-real-valued function on $\mathfrak{X}^{(0)}\mathfrak{Y}$ with the properties* (1) *$f$ is measurable on $\mathfrak{X}^{(0)}$ for each $y$, and* (2) *$f$ is continuous in each argument $y_j$, either on the right or on the left, when the other variables are fixed. Then if $g(x)$ on $\mathfrak{X}^{(0)}$ to $\mathfrak{Y}$ is measurable on $\mathfrak{X}^{(0)}$, so is the function $f(x, g(x))$.*

We take first the case $m=1$, and assume (to fix the ideas) that $f$ is continuous on the left in $y$. We construct a sequence $\{\pi_n\}$ of partitions of the $y$-axis, for example by taking the division points in $\pi_n$ to be

$$l_{ni} = \frac{i}{n}, \qquad (i = -\infty, \cdots, +\infty).$$

Then the set $\mathfrak{X}^{(ni)}$ of points of the measurable set $\mathfrak{X}^{(0)}$ for which $l_{ni} \leqq g(x) < l_n, i+1$, is measurable, and we have

$$\sum_i \mathfrak{X}^{(ni)} = \mathfrak{X}^{(0)}$$

for every $n$. We construct a sequence $\{g_n(x)\}$ of functions measurable on $\mathfrak{X}^{(0)}$ and approaching $g(x)$ from the left by setting $g_n(x) = l_{ni}$ on the set $\mathfrak{X}^{(ni)}$. Hence the function $f(x, g_n(x))$, which equals $f(x, l_{ni})$ on the set $\mathfrak{X}^{(ni)}$, is measurable on $\mathfrak{X}^{(ni)}$, and therefore measurable on $\mathfrak{X}^{(0)}$. Since $f$ is continuous on the left in $y$, we have $\lim f(x, g_n(x)) = f(x, g(x))$, and the last named function is also measurable on $\mathfrak{X}^{(0)}$.

We complete the proof by induction. By the theorem for $m$, $f(x, g(x), y_{m+1})$ is measurable on $\mathfrak{X}^{(0)}$ and continuous

(right or left) in $y_{m+1}$. Hence, by the proof just given, $f(x, g(x), g_{m+1}(x))$ is measurable.

THEOREM II. *Let the set $\mathfrak{W}^{(0)}$ and the function $f(x, y)$ single-real-valued on $\mathfrak{W}^{(0)}$ be such that* (1) *for each $y$, $f$ is measurable in $x$ on every measurable set contained in $\mathfrak{X}^{(y)}$, and* (2) *for each $x$, $f$ is continuous on $y$ in $\mathfrak{Y}^{(x)}$. Let $\mathfrak{X}^{(0)}$ be a measurable set, and let $g(x)$ be a function on $\mathfrak{X}^{(0)}$ to $\mathfrak{Y}$, which is measurable on $\mathfrak{X}^{(0)}$, and such that for a fixed positive number $a$, the neighborhood $[g(x)]_a$ is in $\mathfrak{Y}^{(x)}$ for every $x$. Then the function $f(x, g(x))$ is measurable on $\mathfrak{X}^{(0)}$.*

Divide the space $\mathfrak{Y}$ into a denumerable infinity of "cubes" $\mathfrak{Y}^{(j)}$, with edges parallel to the axes of the space, and maximum diameter less than or equal to the number $a$. Let $\mathfrak{X}^{(j)}$ be the subset of $\mathfrak{X}^{(0)}$ on which $g(x)$ is in the set $\mathfrak{Y}^{(j)}$. Then each $\mathfrak{X}^{(j)}$ is measurable, being a product of measurable sets, and $\sum \mathfrak{X}^{(j)} = \mathfrak{X}^{(0)}$. We consider hereafter only those values of $j$ for which $\mathfrak{X}^{(j)}$ is not empty. Let $y^{(j)}$ be the center of the "cube" $\mathfrak{Y}^{(j)}$. Then for every $x$ in $\mathfrak{X}^{(j)}$, the point $g(x)$ is contained in the closed neighborhood $[y^{(j)}]_b$ (where $2b = a$), and the neighborhood $[y^{(j)}]_b$ in turn is contained in the neighborhood $[g(x)]_a$ and hence in the set $\mathfrak{Y}^{(x)}$. We can now define a function $F(x, y)$ on $\mathfrak{X}^{(j)}\mathfrak{Y}$, equal to $f(x, y)$ on $\mathfrak{X}^{(j)}[y^{(j)}]_b$, measurable on $\mathfrak{X}^{(j)}$ for every $y$, and continuous on $\mathfrak{Y}$ for every $x$. E. g., for points $y$ not in $[y^{(j)}]_b$, set $F(x, y) = f(x, y^{(j)} + c(y - y^{(j)}))$, where $b = c \times$ distance from $y$ to $y^{(j)}$. Then by Theorem I, $F(x, g(x)) = f(x, g(x))$ is measurable on the set $\mathfrak{X}^{(j)}$. Hence $f(x, g(x))$ is measurable on $\mathfrak{X}^{(0)}$.

In the proof of Theorem III, we shall need the following preliminary theorem. (We now drop the abbreviated notation of the preceding paragraphs.)

*Suppose the single-real-valued function $f(x, y)$ is summable on the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$. Then there exists a set $\mathfrak{E}$ of points of the interval $(a, b)$ such that:*

(1) *measure of $\mathfrak{E} = b - a$;*

(2) *the integral*

$$\int_c^y f(x, y)dy = g(x, y)$$

*exists for every $x$ in the set $\mathfrak{E}$ and every $y$ in $(c, d)$;*

(3) *$g(x, y)$ is measurable in $x$ on $\mathfrak{E}$, for every $y$;*

(4) *$|g(x, y)| \leqq M(x)$ for every $y$, where $M(x)$ is summable on $\mathfrak{E}$.*

When we take $y=d$, the statements of this theorem are at least implicitly contained in every treatment of the reduction of a double integral of a summable function to two successive simple integrals.* We obtain the theorem stated for a value $y=y_0<d$ by replacing $f(x, y)$ by a function $f_0(x, y)$, equal to $f$ for $c \leqq y \leqq y_0$, and zero for $y_0<y \leqq d$. It is readily seen that the set $\mathfrak{E}$ effective for $y=d$ is effective for all values of $y$. To obtain the fourth conclusion, we have

$$|g(x,y)| \leqq \int_c^y |f(x,y)|\,dy \leqq \int_c^y |f(x,y)|\,dy.$$

THEOREM III. *Suppose the function $f(x, y)$ is summable on the square $a \leqq x \leqq b$, $a \leqq y \leqq b$. Then there exists a set $\mathfrak{E}$ of points of $(a, b)$, whose measure is $(b-a)$, such that the function*

$$\int_a^x f(x,y)dy$$

*is summable on $\mathfrak{E}$.*

By our preliminary theorem, the function

$$g(x,y) = \int_a^y f(x,y)dy$$

is measurable in $x$ on a set $\mathfrak{E}$ with the specified properties, and satisfies the condition $|g(x, y)| \leqq M(x)$, where $M(x)$ is summable on $\mathfrak{E}$. It is also continuous in $y$ on $(a, b)$. Hence we can extend the range of definition of the function $g(x, y)$ so that the conditions of Theorem I will be satisfied. This, with the inequality $|g(x, x)| \leqq M(x)$, shows that $g(x, x)$ is summable on $\mathfrak{E}$.

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* See de la Vallée Poussin, *Intégrales de Lebesgue*, pp. 50–53; or BULLETIN DE L'ACADÉMIE DE BELGIQUE, Sciences, 1910, p. 768.

Various modifications of Theorem III may easily be secured. For example, in case we make the additional assumptions that the function $f(x, y)$ is bounded and is measurable in $y$ for each $x$, then the set $\mathfrak{E}$ may be replaced by the interval $(a, b)$. These additional assumptions are fulfilled in particular if $f$ is bounded and Borel measurable on the square where it is defined. In this case the function $g(x, x)$ is Borel measurable on $(a, b)$. As another modification we may substitute for the square $a \leqq x \leqq b$, $a \leqq y \leqq b$, a bounded measurable set $\mathfrak{E}_0\mathfrak{E}_0$, consisting of those points of the plane having $x$ and $y$ each in a linear measurable set $\mathfrak{E}_0$. Then the integral is understood to be taken over those points of the interval $(a, x)$ contained in $\mathfrak{E}_0$.

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## A GENERAL THEORY OF REPRESENTATION OF FINITE OPERATIONS AND RELATIONS*

BY B. A. BERNSTEIN

Let $a \bmod n$ denote the least positive residue modulo $n$ of an integer $a$, i. e., the least positive integer obtained from $a$ by rejecting multiples of $n$. Consider the polynomials modulo a prime $p$

$$(1) \qquad a_0 + a_1 x + \cdots + a_{p-1}x^{p-1}, \mod p,$$

$$(2) \qquad f_0(x) + f_1(x)y + \cdots + f_{p-1}(x)y^{p-1}, \mod p,$$

where in (1) $a_i$ are least positive $p$-residues and $x$ ranges over the complete system of least positive $p$-residues, and where (2) is a polynomial modulo $p$ in $y$ whose coefficients $f_i(x)$ are modular polynomials in $x$ of form (1). In a previous paper† I developed a theory of representation of abstract

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† PROCEEDINGS OF THE INTERNATIONAL MATHEMATICAL CONGRESS, TORONTO, 1924.

binary operations and dyadic relations in a finite class of elements, in which theory the polynomials (2) entered fundamentally. I now wish to note the fact that this theory can be extended to finite operations and relations in general.

An $m$-ary operation $O$ in a class $K$ is a rule which determines for every ordered set of $m$ $K$-elements $x_1, x_2, \cdots, x_m$ what $K$-element $O(x_1, x_2, \cdots, x_m)$, if any, corresponds to the set. If $K$ is finite, such a rule may always be given by an $m$-dimensional operation table or, when $m>1$, by $n$ tables each of $m-1$ dimensions. An $m$-adic relation $R$ in $K$ is a rule which states for every ordered set of $m$ $K$-elements $x_1, x_2, \cdots, x_m$ whether or not these elements should be associated together in a proposition $R(x_1, x_2, \cdots, x_m)$. When $K$ is finite, such a rule may always be given by an $m$-dimensional relation table or by $n$ tables each of $m-1$ dimensions ($m>1$), in which the fact that $R(x_1, x_2, \cdots, x_m)$ holds may be indicated by "$+$" and that it does not hold by "$-$".* The function fundamental in the representation of finite $m$-ary operations and $m$-adic relations in general is the modular polynomial $f(x_1, x_2, \cdots, x_m)$ of the form

$$f_0 + f_1 x_m + \cdots + f_{p-1} x_m^{p-1}, \mod p, \tag{3}$$

where $p$ is prime and the coefficients $f_i$ are polynomials modulo $p$ in the $m-1$ arguments $x_1, x_2, \cdots, x_{m-1}$. Noting that when the $K$-elements are $n$ in number they may be labeled $0, 1, \cdots, n-1$, we may state the general theory by the propositions $A$, $B$, $C$ following.

PROPOSITION A. Given an arbitrary set of least positive $n$-residues

$$e_0, e_1, \cdots, e_{n-1}; \tag{4}$$

if and only if $n$ is prime, a function $f(x)$ of form (1) can always be obtained such that

$$f(0) = e_0, \quad f(1) = e_1, \quad \cdots, \quad f(n-1) = e_{n-1}; \tag{5}$$

namely (1) in which (modulo $n$)

* This "$\perp$" notation has been used by H. M. Sheffer.

$$(6)\qquad a_0 = e_0,\ a_1 = \frac{\begin{vmatrix} e_1 - e_0 & 1 & \cdots & 1 \\ e_2 - e_0 & 2^2 & \cdots & 2^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ e_{n-1} - e_0 & (n-1)^2 & \cdots & (n-1)^{n-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ n-1 & (n-1)^2 & \cdots & (n-1)^{n-1} \end{vmatrix}},$$

$$\cdots,\ a_{n-1} = \frac{\begin{vmatrix} 1 & 1 & \cdots & e_1 - e_0 \\ 2 & 2^2 & \cdots & e_2 - e_0 \\ \cdots & \cdots & \cdots & \cdots \\ n-1 & (n-1)^2 & \cdots & e_{n-1} - e_0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ n-1 & (n-1)^2 & \cdots & (n-1)^{n-1} \end{vmatrix}}.$$

PROPOSITION B. Given any $m$-dimensional operation table for the $K$-elements $0, 1, \cdots, n-1$; a function equivalent to this table may be found having the form

$$(7)\qquad f + 0/\varphi,$$

where $f$ and $\varphi$ are modular polynomials of form (3), with the $x$'s ranging over the $K$-elements.

PROPOSITION C. Given any $m$-dimensional relation table for the $K$-elements $0, 1, \cdots, n-1$; an equation equivalent to this table may be found having the form

$$(8)\qquad f = 0,$$

where $f$ is a modular polynomial of form (3), with the $x$'s ranging over the $K$-elements.

For the proof of Proposition A the reader is referred to the paper cited above.

The second term of (7) is designed to care for operations that do not satisfy the conditions of closure. To see the

truth of Proposition B, consider several cases. (i) If $n$ is prime and the given operation satisfies the condition of closure, function (7) is the polynomial of form (3) obtained from the given table by repeated application of Proposition A. (ii) If $n$ is prime and the given operation does not satisfy the closure condition, consider the table got from the given table by assigning $K$-elements to the sequences $x_1, x_2, \cdots, x_m$, to which no $K$-element corresponds; consider also the table got from the given table by assigning **0** to each sequence to which no $K$-element corresponds and a $K$-element not **0** to any other sequence. The polynomials (3) equivalent to the derived tables will be respectively the $f$ and $\varphi$ of (7). (iii) If $n$ is composite, consider any operation table for a prime number $p(>n)$ of elements which will give the original table when the $x$'s range over the $n$ elements of $K$; the function equivalent to this table, with the $x$'s ranging over the $K$-elements, will be the required function (7).

To see that Proposition C is true, consider any operation table got from the given relation table by changing each "$+$" to **0** and each "$-$" to some $K$-element not **0**; the function equivalent to this operation table will be the $f$ of (8).

Our theory of representation shows that any finite mathematical system, quantitative or non-quantitative, can be represented arithmetically (and geometrically). The theory also makes clear the nature of operations and relations, and it brings out the fact that an $m$-ary operation is the same as an $(m+1)$-adic relation, and $m$-adic relation the same as one or more $(m-1)$-ary operations $(m>1)$.*

UNIVERSITY OF CALIFORNIA

* For other applications and for various illustrations, see the above cited paper. See also this BULLETIN, vol. 30 (1924), p. 24, and AMERICAN JOURNAL, vol. 46 (1924), p. 110.

# ISOLATED SINGULAR POINTS OF HARMONIC FUNCTIONS*

BY G. E. RAYNOR

Bôcher has given several theorems relating to the nature of a harmonic function in the neighborhood of an isolated singular point.† E. Picard‡ has proved two of the theorems given by Bôcher in the paper mentioned above. In both papers the following fundamental theorem occurs.

*If a function $f(x, y)$ is continuous and harmonic everywhere in the interior of a closed plane region with the exception of an isolated point $P$ in the neighborhood of which $f(x, y)$ tends to plus infinity for every mode of approach to $P$, $f$ is of the form $c \log (1/r)+V$ in this neighborhood, where $c$ is a positive constant, $r$ the distance from $(x, y)$ to $P$ and $V$ is a function harmonic everywhere in the neighborhood of $P$, including $P$ itself.*

The analogous theorem for three-space is also proved in each paper.

Picard's proof for the plane makes use of complex variables. However, as Bôcher points out, it is desirable to have an independent proof in order to follow out Riemann's idea of basing the theory of complex variables on the theory of real harmonic functions in two variables. Bôcher's discussion is general and applies to harmonic functions in any number of variables. However, both Bôcher and Picard, in the case of the theorem cited for three-space, apply Green's Formula to a region bounded partly by the surface

$$f(x, y, z) = \text{constant}.$$

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* Presented to the Society, October 31, 1925. See also a paper by O. D. Kellogg, *On some theorems of Bôcher concerning isolated singular points of harmonic functions*, which is to appear in the next issue of the BULLETIN.

† This BULLETIN, vol. 9 (1902–3), p. 455 ff.

‡ BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 52 (1924).

Now this involves an integral over this surface and furthermore the integral contains the normal derivative of $f$ on the surface. In order to make the discussion complete it would seem that some consideration of the nature of the surface $f=c$ is necessary in order to know if $df/dn$ has a meaning and if the integral exists. Such a discussion is of course possible but by no means simple. See, for example, the treatment of the analogous problem for Green's Function in the plane in Osgood's *Funktionentheorie.** It is the purpose of the present paper to obtain for the plane additional results, as well as those of Bôcher, but by a different method. Furthermore, most of the present discussion can be generalized at once to three-space.

We suppose, then, that $f(x, y)$ is a function harmonic everywhere in a plane region $A$ except possibly at a single interior point $P$. We wish to study the nature of $f$ in the vicinity of $P$.

About $P$ as a center, and with radius $r_1$, draw a circle $C_1$ which lies entirely in the region $A$. Also about $P$ draw a second circle $C_2$ with radius $r_2<r_1$. Let $V'$ be the function which is harmonic everywhere in $C_1$ and which on $C_1$ coincides with $f$. This function we know exists and is given by Poisson's integral. In the following we shall indicate the region bounded by two concentric circles $C_i$ and $C_j$ by the symbol $C_iC_j$.

Now the difference

$$f-V'\equiv F \tag{1}$$

is continuous and harmonic in $C_1C_2$ and is zero everywhere on $C_1$. Let $r$ be the distance of any interior point $(x, y)$ of $C_1$ to $P$. Then the function

$$\log\frac{1}{r}-\log\frac{1}{r_1}\equiv W(r)$$

is evidently also harmonic in $C_1$ $C_2$ and takes the value

* First edition, p. 588; 2d ed., p. 673.

zero on $C_1$. Since $V'$ is given by Poisson's integral, we know that on $C_1$, $dV'/dn$ exists and is continuous. Also since $f$ is harmonic *on* $C_1$ its normal derivative also exists and is continuous there. Hence the same is true of the difference $F \equiv f - V'$. The same is seen directly to be true of $dW/dn$. Also, since both $W$ and $F$ are harmonic *on* $C_2$ their normal derivatives exist on this circle.

Let us now apply Green's Formula to the region bounded by $C_1$ and $C_2$. We have

$$\int_{C_1}\left[F\frac{dW}{dn} - W\frac{dF}{dn}\right]ds = 0,$$

where the normal derivatives are taken toward the interior of the region $C_1\,C_2$. But on $C_1$, $F$ and $W$ are both zero, and hence

$$\int_{C_1}\left[F\frac{dW}{dn} - W\frac{dF}{dn}\right]ds = 0, \tag{2}$$

or

$$\int_{C_2} F\frac{dW}{dn}\,ds = \int_{C_2} W\frac{dF}{dn}\,ds. \tag{3}$$

But along $C_2$ we have $dW/dn = -1/r$ which is constant on $C_2$ as is also $W$. Hence we have, substituting in (3),

$$\int_{C_2} F ds = -r_2\ W(r_2)\int_{C_2}\frac{dF}{dn}\,ds. \tag{4}$$

But by a well known theorem on harmonic functions

$$\int_{C_1C_2}\frac{dF}{dn}\,ds = \int_{C_1}\frac{dF}{dn}\,ds + \int_{C_1}\frac{dF}{dn}\,ds = 0,$$

and hence (4) can be written

$$\int_{C_2} F ds = r_2 W(r_2)\int_{C_1}\frac{dF}{dn}\,ds. \tag{5}$$

Let us set

$$\frac{1}{2\pi}\int_{C_1}\frac{dF}{dn}\,ds = c.$$

We have then from (5) the result that the function $F$ satisfies the integral equation

$$\int_{C_2} F\,ds = 2\pi r_2\left(\log\frac{1}{r_2} - \log\frac{1}{r_1}\right) \tag{6}$$

where $C_2$ is any circle with $P$ as center, and radius $r_2 < r_1$.

Now the general solution of (6) is easily seen to be

$$F = c\left(\log\frac{1}{r_2} - \log\frac{1}{r_1}\right) + \Phi(x,y),$$

where

$$\int_{C_2}\Phi(x,y)\,ds = 0 \tag{7}$$

and for our purposes $\Phi$ must be harmonic in $C$. Substituting in (1) we have

$$\begin{aligned} f &= c\left(\log\frac{1}{r} - \log\frac{1}{r_1}\right) + \Phi(x,y) + V' \\ &= c\log\frac{1}{r} + \Phi(x,y) + V, \end{aligned} \tag{8}$$

where $V = V' - c\log(1/r_1)$. Since the form of $f$ is of course independent of any circle as $C_2$ the form (8) holds at any interior point of $C_1$ except $P$. Furthermore, on $C_1$, $\Phi \equiv 0$.

Point $P$ will in general be a singular point of $\Phi$, and we shall show as a matter of fact, unless $\Phi(x,y) \equiv 0$, that in the neighborhood of $P$ the function $\Phi$ will tend toward both plus infinity and minus infinity, that is, that there will exist a mode of approach to $P$ along which $\Phi$ will tend toward plus infinity and *also* a mode of approach to $P$ for which it will tend to minus infinity. Furthermore, the same statement will be seen to be true concerning the sum

$$c\log\frac{1}{r} + \Phi.$$

To prove this, let us write $\Phi$ in the form

$$\Phi = \frac{1}{r}\Psi(x,y). \tag{9}$$

It is clear that in the open region bounded by $C_1$ and $P$ the function $\Psi(x, y)$ is continuous. We shall now define two functions $f_1(x, y)$ and $f_2(x, y)$ in this open region as follows:

$$\begin{aligned}
f_1(x,y) &= \Psi(x,y) \text{ at points where } \Psi(x,y) \geqq 0,\\
f_1(x,y) &= 0 \quad " \quad " \quad " \quad \Psi(x,y) < 0,\\
f_2(x,y) &= \Psi(x,y) " \quad " \quad " \quad \Psi(x,y) \leqq 0,\\
f_2(x,y) &= 0 \quad " \quad " \quad " \quad \Psi(x,y) > 0.
\end{aligned}$$

From the continuity of $\Psi(x, y)$ it is clear that $f_1$ and $f_2$ are continuous in their region of definition. We have thus

$$\Psi(x,y) = f_1(x,y) + f_2(x,y). \tag{10}$$

From (7), since $1/r$ is constant on any circle with $P$ as center we have

$$\int_C \Psi(x,y)ds = 0,$$

where $C$ is such a circle interior to $C_1$. Hence

$$\int_C f_1(x,y)ds + \int_C f_2(x,y)ds = 0 \tag{11}$$

or in polar coordinates, with $P$ as pole, and any fixed line through $P$ as polar axis,

$$\int_0^{2\pi} f_1 d\theta + \int_0^{2\pi} f_2 d\theta = 0 \tag{12}$$

for any value of $r$. Consider now an arbitrary circle $C_2$ with $P$ as center and radius $r_2 < r_1$. The expression*

$$U(x,y) = \frac{1}{2\pi r_2}\int_{C_2}\Phi ds - \frac{1}{\pi}\int_{C_2}\Phi\frac{\cos\phi}{d}ds \tag{13}$$

* Goursat, *Cours d'Analyse Mathématique*, vol. 3, p. 222.

defines a function $U$ which is harmonic at any point $(x, y)$ exterior to $C_3$ and is such that as the point $(x, y)$ approaches a point on $C_3$, $U(x, y)$ approaches the value of $\Phi$ at that point. In the above integrals $d$ is the distance from $(x, y)$ to a variable point $M$ on $C$, and $\phi$ the angle betwen $d$ and the radius of $C_3$ drawn to $M$.

By (7) the first of these integrals is zero, and hence

$$U(x,y) = -\frac{1}{\pi}\int_{C_3} \Phi \frac{\cos\phi}{d}\, ds = -\frac{1}{\pi}\int_0^{2\pi} \Psi \frac{\cos\phi}{d}\, d\theta.$$

Substituting the value of $\Psi$ from (10) we obtain

$$(14)\quad U(x,y) = -\frac{1}{\pi}\int_0^{2\pi} f_1 \frac{\cos\phi}{d}\, d\theta - \frac{1}{\pi}\int_0^{2\pi} f_2 \frac{\cos\phi}{d}\, d\theta.$$

But $f_1$ and $f_2$ are of constant sign and hence applying the first theorem of the mean for integrals to (14) we have

$$(15)\quad U(x,y) = -\frac{1}{\pi}\left\{\left[m_1 \frac{\cos\phi}{d}\right]\int_0^{2\pi} f_1\, d\theta + \left[m_2 \frac{\cos\phi}{d}\right]\int_0^{2\pi} f_2\, d\theta\right\}$$

where $[m \cos(\phi/d)]$ indicates a mean of $\cos(\phi/d)$. But by (12), (15) becomes

$$U(x,y) = -\frac{1}{\pi}\left\{\left[m_1 \frac{\cos\phi}{d}\right] - \left[m_2 \frac{\cos\phi}{d}\right]\right\}\int_0^{2\pi} f_1\, d\theta.$$

Let $\bar{f}_1(r_3, \theta)$ indicate the maximum value of $f_1$ on $C_3$. Then

$$(16)\quad |U(x,y)| \leqq 2\left|\left[m_1 \frac{\cos\phi}{d}\right] - \left[m_2 \frac{\cos\phi}{d}\right]\right| \bar{f}_1(r_3,\theta).$$

Suppose now that $\bar{f}_1$ approaches zero as $r_3$ approaches zero. Then, since for any point on $C_1$, $[m\cos(\phi/d)]$ is bounded, $|U(x, y)|$ can be made less than $\epsilon/4$ at all points on $C_1$ by taking $r_3$ sufficiently small, $r_3 = r_3'$, say. Now on $C_3$, $\Phi$ and $U$ are equal, while on $C_1$, $\Phi = 0$. Hence in the region $C_1C_3$,

$\Phi$ will differ from $U$ by less than $\epsilon/2$. Now consider any point $(a, b)$ except $P$, interior to $C_1$. We can take $r_3$ so small that $(a, b)$ lies in $C_1C_3$. Also by (16), $r_3$ can be taken so small, $r_3 = r_3''$ say, that $U(a, b) < \epsilon/2$. Hence by taking $r_3$ equal to the smaller of $r_3'$ and $r_3''$ we have

$$|\Psi(a,b)| < \epsilon.$$

Since $\epsilon$ is any arbitrarily small number and $(a, b)$ is any point in $C_1$ except $P$, we have the result that if $\overline{f_1}$ approaches zero as $r_3$ approaches zero, $\Psi$ is identically zero in $C_1$.

Therefore, unless $\Phi \equiv 0$, there must be a mode of approach to $P$ for which $f_1$ remains greater than some positive constant $c_1$, or in other words for which $\Phi$ remains greater than $c_1/r$. In the same manner we can prove that there exists a mode of approach to $P$ for which $\Phi$ remains less than $-c_2/r$ where $c_2$ is a positive constant.

Suppose the constant $c$ in (8) is positive. It is easy to prove that

$$\lim_{r \to 0} \left[ c \log \frac{1}{r} - \frac{c_2}{r} \right] = -\infty$$

and hence for some mode of approach to $P$

$$\lim_{r \to 0} \left[ c \log \frac{1}{r} + \Phi(x,y) \right] = -\infty.$$

In the same way, if $c$ is negative, for some mode of approach to $P$

$$\lim_{r \to 0} \left[ c \log \frac{1}{r} + \Phi(x,y) \right] = +\infty.$$

We can now formulate the following theorems.

THEOREM I. *In the neighborhood of an isolated singular point $P$, a harmonic function $f(x, y)$ is of the form*

$$f(x,y) = c \log \frac{1}{r} + \Phi(x,y) + V(x,y)$$

*where $c$ is a constant, $r$ the distance from $(x, y)$ to $P$, $\Phi(x, y)$, unless it is identically zero, harmonic in the neighborhood of $P$ and such that there exist modes of approach to $P$ for which the sum $c \log (1/r)+\Phi$ tends towards plus infinity and also toward minus infinity; and $V$ is harmonic everywhere in the neighborhood of $P$, including $P$.*

THEOREM II. *If a harmonic function $f(x, y)$ is bounded in the deleted neighborhood of an isolated singular point, it is in this neighborhood equal to a function which is harmonic in this neighborhood and also at $P$, and the singularity can be removed by defining $f(P)$ to be equal to* $\lim f(x, y)$ *as $(x, y)$ approaches $P$.*

For the constant $c$ and the function $\Phi$ must be identically equal to zero if $f$ is bounded and we have $f = V$ in the neighborhood of $P$.

THEOREM III. *If a harmonic function tends toward plus infinity (minus infinity) in the neighborhood of an isolated singular point, it is of the form $f = c \log (1/r) + V$ where $c$ is a positive (negative) constant and $V$ is harmonic everywhere in this neighborhood, including $P$.*

For by Theorem I, $\Phi$ must be identically zero in this case.

THEOREM IV. *A harmonic function $f(x, y)$ cannot have an isolated singular point $P$ which is such that for certain modes of approach to $P$, $f(x, y)$ tends toward plus infinity (minus infinity) and remains finite for all other modes of approach to $P$.*

For if $\Phi(x, y) \equiv 0$, we have the case described in Theorem III, while if $\Phi(x, y)$ is not identically zero, Theorem IV follows from Theorem I.

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# A LIST OF ERRORS IN TABLES OF THE PELL EQUATION

BY D. H. LEHMER

Five tables of solutions of the Pell equations

$$x^2 - Ay^2 = 1, \tag{1}$$

$$x^2 - Ay^2 = -1, \tag{2}$$

have been published giving solutions* from $A=2$ to $A=1700$. Arranged in chronological order these tables are as follows:

1. A. M. Legendre, *Théorie des Nombres*, Paris, 1798, Table xii.
2. C. F. Degen, *Canon Pellianus etc.*, Copenhagen, 1817.
3. A. M. Legendre, *Théorie des Nombres*, 3d ed., 1830, Table x.
4. C. E. Bickmore, BRITISH ASSOCIATION REPORT, vol. 53 (1893), p. 73.
5. E. E. Whitford, *The Pell Equation*, New York, 1912, p. 102.

The scope of each of the tables is as follows:

1, 3 Legendre, $A=2$ to 1003. Gives solutions of (2) when possible, otherwise of (1).

2. Degen, $A=2$ to 1000. Gives solutions of (1) and also of (2) when possible.

4. Bickmore, $A=1001$ to 1500. Gives solutions of (2) when possible, otherwise of (1).

5. Whitford, $A=1501$ to 1700. Gives solutions of (1) and also of (2) when possible.

A detailed account of these tables together with a list of errors in Legendre's and Degen's tables is included in an article by A. J. C. Cunningham.† This list, obtained

* In the present paper, by "solution" we mean "fundamental solution."

† MESSENGER OF MATHEMATICS, vol. 46, p. 49.

evidently by reading the tables together, is incomplete and not altogether correct.

The five tables mentioned above have been examined and thoroughly checked by the present writer. It is believed that no error has escaped detection.

The ordinary method of verifying a solution consists in actually substituting the quantities $x$ and $y$ in the equation. Cayley remarks that it is easier to calculate $Ay^2 \pm 1$ and then show that the square root of this quantity is actually $x$. Although this method is very laborious when $x$ and $y$ are large, one would naturally suppose that in using this check all errors would be detected. This however seems not to be the case. Non-typographical errors appear in the tables of Bickmore and Whitford though both tables were submitted to this check before publication. It is significant that these errors occur in solutions which are unusually large.

In obtaining the following list of errors a different method of checking was used in the case of solutions consisting of more than five or six digits. Instead of substituting the actual values of $x$ and $y$ in the equation, their remainders on division by some modulus were used. Three different moduli were employed, namely, 1001, 10001, and 100001, depending on the printing in the table to be checked. These moduli, chosen for their ease in casting out, made the checking extremely rapid and practically independent of the size of the solution. The necessary calculations were made on an ordinary computing machine without putting pen to paper.

Although this method does not afford as complete a check as would be obtained by using the actual values of $x$ and $y$, the chance of a compensating error is certainly negligible when using these large moduli.

In order to save space and avoid confusion only the corrected entries corresponding to arguments in which errors have been made are given below. Also uniform notation is used, the equation being taken as

$$x^2 - Ay^2 = \pm 1.$$

There are several points of interest to be noted. In the first edition of Legendre's table for every argument $A<250$ which has an error in $x$, the same error is made in the $x$ corresponding to the argument $4A$. This is explained as follows. If in solving the equation for the argument $A$, the value for $y$ is found to be even $(2n)$ we can write at once

$$x^2-A(2n)^2=x^2-4An^2=\pm 1.$$

The solution for $4A$ was therefore obtained by merely dividing the $y$ by 2 and using the original $x$. This division by 2 was overlooked in the case of $A=344$, the entry given being identical with that for $A=86$. In general it can be shown that the solution for the argument $p^2A$ can be obtained from a fundamental or multiple solution for the argument $A$. This theorem seems not to have been noticed or at least not used to advantage by the workers on the Pell equation. For numbers containing a square factor greater than unity (there are 627 such $<1700$) the expansion of $\sqrt{A}$ in a continued fraction and the calculation of the successive convergents could have been avoided entirely.

Before publishing the 3d edition of Legendre's table, the table in the first edition was compared with that of Degen. Nevertheless one error was carried over from the first edition and two from Degen's table. The other errors are for the most part typographical. There are two unimportant errors not listed below in Bickmore's table. For $A=1051$ the values of $x$ and $y$ are interchanged and the argument $A=1361$ should be printed with an asterisk (1361*) to designate equation (2). There are several errors of little consequence in the introductory table. Whitford's table contains one insignificant error; the argument 1589 is printed 1599. In Appendix A of Whitford's book are given the partial quotients and denominators of the complete quotients occurring in the expansion of $\sqrt{A}$ in a continued fraction from $A=1501$ to $A=2012$. In preparing a table

of solutions of the Pell equation for $A=1701$ to $A=2000$, the present writer has found the following errors in the former table.

| $A$ | |
|---|---|
| 1733* | The 6th partial quotient should be 3 and not 2. |
| 1822 | The 23d partial quotient and denominator of the 23d complete quotient are missing. They are 1 and 54 respectively. |
| 1852 | The 29th partial quotient should be 20 and not 16. |
| 1963 | The entry here should be: |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 44 | 3 | 3 | 1 | 2 | 3 | 2 | 29 | 9 | 1 | 4 | (3) |
| | 27 | 22 | 51 | 29 | 23 | 38 | 3 | 9 | 66 | 17 | (26) |

The above list comprises all the $A$'s between 1700 and 2000 for which there is an error in a partial quotient.

It should be stated that in the tables of Degen, Bickmore, and Whitford which give the above mentioned elements of the expansion of $\sqrt{A}$, $A<1701$, only the solutions of the Pell equation were checked.

1. Legendre, 1st edition, Table xii.

| $A$ | |
|---|---|
| 133 | $x=2588599$ |
| 214 | $x=695359189925$ |
| | $y=47533775646$ |
| 236 | $x=561799$ |
| 301 | $y=339113108232$ |
| 307 | $x=88529282$ |
| 331 | $x=2785589801443970$ |
| 343 | $x=130576328$ |
| | $y=7050459$ |
| 344 | $y=561$ |
| 355 | $y=50676$ |
| 365 | $x=3458$ |
| 397 | $x=20478302982$ |
| | $y=1027776565$ |
| 526 | $x=84056091546952933775$ |
| | $y=36650197573242955532$ |
| 532 | $x=2588599$ |
| 613 | $x=481673579088618$ |

* This error was also found by Cunningham, loc. cit.

## 1. Legendre, 1st edition—*continued*

*A*

619 $x = 517213510553282930$
$y = 20788566180548739$
629 $x = 7850$
655 $x = 737709209$
$y = 28824684$
664 $y = 66007821$
673 $x = 48813455293932$
694 $x = 38782105445014642382885$
$y = 14721485909039976721 14$
718 $x = 8933399183036079503$
722 $x = 22619537$
$y = 841812$
753 $y = 11243313484$
771 $x = 2989136930$
$y = 107651137$
801 $x = 500002000001$
$y = 17666702000$
806 $x = 6166395$
809 $x = 433852026040$
$y = 15253424933$
833 $x = 9478657$
851 $x = 8418574$
856 $x = 695359189925$
$y = 23766887823$
865 $x = 348345108$
871 $x = 19442812076$
$y = 658794555$
878 $x = 9314703$
$y = 314356$
886 $y = 260148796464024194850378$
944 $x = 561799$
965 $x = 14942$
$y = 481$
995 $x = 8835999$
1001 $x = 1060905$

## 2. Degen, Canon Pellianus, Table 1.

*A*

238 $y = 756$
277 $x = 159150073798980475849$
421 $y = 1890739959518390208804997807062 60$
437 $x = 4599$
613 $y = 1874154578483199788030878434 0$
641 $x = 2609429220845977814049$
$y = 103066257550962737720$

### 2. Degen, Canon Pellianus—*continued*

*A*

653 $x = 10499986568677299849$
672 $x = 337$
751 $x = 729331846679488242441896O$
823 $x = 235170474903644006168$
919 $y = 14783444239653675978149958 9$
945 $x = 275561$
949 $y = 19789181711517243032971740$
951 $x = 224208076$

### 3. Legendre, 3d and 4th editions, Table x.

94 $x = 2143295$
116 $x = 9801$
149 $y = 9305$
271 $x = 115974983600$
308 $x = 351$
479 $y = 136591$
629 $x = 7850$
667 $y = 4147668$
749 $x = 1084616384895$
751 $x = 729331846679488242441896O$
809 $x = 433852026040$
823 $x = 235170474903644006168$
1001 $x = 1060905$

### 4. Bickmore, British Association Report, 1893.

1014 $y = 146246$
1366 $y = 61\ 98787\ 71121\ 28467\ 93128\ 64853\ 64042$

### 5. Whitford, The Pell Equation.

1549 $y = 12223\ 09542\ 82674\ 74959\ 34242\ 68334\ 63805—$
$08818\ 07626\ 31786\ 81966\ 09867\ 28279\ 63220$
1566 $y = 308792110$
1615 $y = 81732$
1669 $y = 572\ 84717\ 32803\ 87374\ 12405\ 68998\ 80229—$
$34138\ 39259\ 82496\ 64340$

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## PRINGSHEIM ON COMPLEX VARIABLES

*Vorlesungen ueber Zahlen- und Funktionenlehre.* Vol. II, pt. 1: *Grundlagen der Theorie der analytischen Funktionen einer komplexen Veraenderlichen.* By Alfred Pringsheim. Leipzig, B. G. Teubner, 1925. xv+626 pp.*

In laying the foundations for analytic functions of a complex variable it is possible to proceed from three practically equivalent points of view, the Meray-Weierstrass, which bases its developments upon the properties of power series, the Cauchy-Riemann point of view, in which the functions to be considered are limited by the existence of a differential quotient, and what might be called the Cauchy-Morera point of view, in which the properties of the function depend upon the properties of the curvilinear integral. Of these, the first two seem to be the only ones which are practicable, and are usually considered simultaneously, each contributing towards the development of a simple and elegant theory. Only occasionally do we find the one emphasized to the almost complete exclusion of the others.

The volume under discussion has for its avowed purpose the development of the foundations of the theory of analytic functions as far as possible along the Meray-Weierstrass lines, i.e., it is based on power series and their properties. This is probably a consistent point of view, in consideration of the fact that this volume is the second in a series, the first of which was devoted to series and sequences of numbers. The choice and arrangement of material is governed then by the desire to accomplish as much as possible by the use of power series.

A brief survey of the contents of the book may be of value.

The first chapter, of an introductory character, is devoted to the real variable and some of the essential properties of functions of real variables. We find, then, discussions of the one to one correspondence between real numbers and points on a line, limits of functions, and properties of continuous functions. The introduction of two variables leads to a consideration of regions and their boundaries, and this in turn to the Jordan Curve Theorem. The method of attack is via the properties of what Pringsheim calls "Treppenpolygone," which one might translate "stair polygons," consisting of segments of straight lines parallel to the coordinate axes, no point on the boundary occurring twice. This section is of a rather more advanced character than most of the remainder of the book. There is further a brief treatment of double and iterated limits and corresponding properties of functions. The chapter closes by indicating as the reason for adopting the power series as the form of function to be considered, its property of being determined throughout its region of existence by its values at an infinity of arbitrarily chosen points, and the desirability of using complex numbers, as giving perhaps a method for continuing power

* Volume I was reviewed for this BULLETIN by C. N. Moore, part 1, in vol. 25 (1919), p. 470; parts 2 and 3, in vol. 28 (1923), pp. 63–5.

series in real variables beyong certain points of divergence. From an elementary point of view neither of these reasons seems convincing. The determination of a function by an infinity of valùes comes rather late in experience, and the fact that the power series is the medium through which this can be accomplished is not exactly self-evident.* The same is true of the notion of continuation as applied to power series. Of course, as indicating two of the important things to be expected of the theory, these two points are well taken.

The second chapter treats in the first place, the complex number, its geometric representation, addition, subtraction, multiplication, and division and their geometric equivalents. The author postpones the introduction of the polar form of the complex number to a later point, so the discussion of multiplication from the geometric aspect is slightly awkward. Functions of a complex variable are introduced and the desirability of considering a special class of these functions indicated. Special consideration from the point of view of transformation is given in the usual fashion to the elementary functions, the linear integral function, the linear fractional function, and the function $y=x^2$ and its inverse.

The third chapter is concerned with rational functions of the complex variable. The integral function is considered first and a proof of the fundamental theorem of algebra given. The greater part of this chapter is then of an algebraic character, including the Lagrange formula of interpolation, the expression of symmetric functions of the roots as rational functions of the coefficients of a polynomial, and the expression of a rational fractional function in terms of partial fractions.

The fourth chapter turns to the consideration of properties of power series. We find first a general introduction to the convergence of sequences and series of functions, including due emphasis on uniform convergence and the Weierstrassian maximal convergence. This leads readily to the properties of the function defined by power series in the interior of the circle of convergence. The behavior of the series in approaching a point of convergence of the series on the circle of convergence is also discussed. In order to obtain the Cauchy theorem on the upper limit of the values of the coefficients of a power series, Pringsheim finds it desirable to introduce the notion of mean value of a function on a circle inside its region of definition. This is based on the possibility of determining the $2^n$th roots of unity by algebraic processes. If $c_n$ is the $2^n$th root of unity with maximum real part and positive imaginary coefficient, then the mean value of the function $f$ on the circle of radius $r$ around the origin is defined as

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^{n-1} f(c_n^k r)$$

* It is an interesting problem (to our knowledge still unsolved) to determine what conditions must be satisfied by the values of a function at a denumerably infinite set of points in a finite region so that they determine an analytic function or a power series, and to determine the form of the function in question.

which limit exists if the function $f$ is continuous on the circle. This mean value is obviously (as Pringsheim points out in the preface) identical with the Cauchy line integral

$$\frac{1}{2\pi i}\int \frac{f(z)dz}{z}$$

along the circle of radius $r$ around the origin. This mean value enables him not only to obtain the limits for the coefficients but also the coefficients of the power series with positive and negative exponents, in terms of the value of the function along a circle in the region of convergence, including also an expression for the power series itself.

Whether the omission of the line integral and its replacement by the mean value is justifiable is open to question. It is true that in power series considerations, the integral around a circle is paramount. Also the trend in modern mathematics seems to be towards the consideration of mean values rather than integrals only. However, it is not entirely clear how the mean value notion can be effectively extended to curves other than circles. Moreover in the line integral, we possess a powerful tool which produces results which do not seem to be possible by other means.

There follow theorems on the rearrangement of series of power series into power series, viz. the Cauchy, Weierstrass and Vitali theorems, and in the proof of the latter, use is made of the properties of the coefficients obtained through the mean value. A natural sequel is the expansion of $f(x+h)$ in powers of $h$, the definition of the derivatives of the function as the coefficients of $h^n/n!$, the properties of derivatives so defined and questions relating to the continuation of power series.

Chapter V is concerned with the definition and properties of monogenic analytic functions. The definition is the usual one, as the totality of power series which can be derived from a given one by analytic continuation. We have the result that such a function can be expanded in the neighborhood of any point interior to the region of existence, i.e. every such point is a regular point. This leads to a consideration of the equivalence of the notion of uniform differentiability (i.e. the existence of the

$$\lim_{h=0}\frac{f(x+h)-f(x)}{h}$$

uniformly) in a neighborhood of a point and regularity. We find then also at this stage the Cauchy-Riemann differential equations and a discussion of their role in the determination of the real and imaginary parts of an analytic function.

Pringsheim justifies his choice of power series as a method of attack on the theory of functions of a complex variable in part by its close relationship to arithmetic expressions (i.e. expressions obtainable from $x$ by the four fundamental operations and limits of sequences). It is the simplest sequel to the rational integral function via the use of the limit process. He holds that the definition of a class of functions by means of a property (in this case the existence of the differential quotient) is not so satisfying.

In this instance, the end justifies the means. On the other hand, one would hardly be tempted to base a theory of continuous functions on arithmetic expressions, to wit, treating them as the functions defined by uniformly convergent sequences of polynomials.

This chapter contains also a proof of the theorem that a function regular in a simply connected region is necessarily single-valued, this being the main use made of the results on Jordan curves of the first chapter. The chapter concludes with the types of possible singularities of analytic functions and a discussion of their characteristics.

The two final chapters of the book are concerned with the elementary transcendental functions and their inverses. The exponential function is derived as the simplest function without singularities and zeros in the finite part of the plane. Its identity with the function

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

is then shown. The sine and cosine functions are defined in terms of the exponential function and their properties deduced mainly from their power series expressions. The infinite product forms of these functions are developed as well as the partial fraction expressions of the quotient functions, which in turn lead to a consideration of the Bernoulli and Euler numbers. The chapter contains also a proof of the transcendental character of $e$ and $\pi$. After a brief general consideration of the inversion of power series at the beginning of the last chapter, the theory thus derived is applied to the logarithmic function and the inverse trigonometric functions. A complete discussion of the nature of the general exponent $b^a$ where $b$ and $a$ are complex numbers, occurs here also.

An appendix containing references for further study, and comments additional to those contained in footnotes concludes the volume.

As an introduction to the study of power series and their possibilities, this volume is admirable. It gives a consistent and excellent treatment of what can be done in the theory of functions of a complex variable from this point of view, though it does occasionally sacrifice in elegance and completeness to attain this end. So while it is an interesting exposition, it does seem unsatisfactory as an introduction to the subject of analytic functions in its omission of the line integral and its elegant and important consequences. Perhaps these are to be taken up in the second part of this volume.

The presentation of the subject matter is excellent and clear, and sins rather on the side of too much rather than too little explanation. As a whole the book seems rather diffuse, there are a number of repetitions, due to the method pursued, and it leaves rather the impression of a compendium on the elementary phases of the subject than an introduction, especially in its inclusion of many topics not usually considered in books on this subject.

T. H. HILDEBRANDT

## WHITE ON CUBIC CURVES

*Plane Curves of the Third Order*. By H. S. White. Published with the cooperation of the National Research Council. Cambridge, Harvard University Press, 1925. xii+168 pp. $2.75.

The author's purpose in writing this book clearly has been to produce, not a detailed treatise on cubic curves, but a clear, readable, and rigorous textbook explaining the outstanding projective properties of these curves. Thus his fundamental problem has been throughout, to select judiciously from a wealth of available material, and to present skillfully, the main features of the theory of cubic curves. In this he has succeeded admirably. The style is nowhere hurried, and the proofs are painstakingly thorough; yet the essentials of the subject are presented in a volume that is by no means lengthy. The author hopes that this work "may be found helpful as a stepping-stone to many extensive and beautiful treatises on special themes." For this purpose a considerable number of references are given.

The book starts with a study of the properties of real cubic curves invariant under real projective transformations. This is followed by a discussion of the simplest theorems on the intersections of cubics with other curves, especially with straight lines. There follows a discussion of considerable length on the properties of the chief invariant loci. Symbolic notation is here introduced and used, but the elaborate formal machinery of this notation is, as far as possible, deliberately and skillfully avoided. Several methods of generating the general cubic are then given, after which the problem of geometry on a cubic is approached, in the most natural and logical manner, by establishing and using Noether's fundamental theorem. This is followed by the parametric representation of the non-singular cubic by means of elliptic functions, with various applications showing the power of this representation. The final chapters bring in the notion of apolarity and point out some of the consequences of this theory.

One who wishes to secure as easily as possible an insight into the spirit and the methods of the projective theory of higher plane curves will find that a preliminary study of cubic curves constitutes an excellent introduction to the field. Most of the principal theories and modes of reasoning are duly exemplified, but in a preliminary and simplified form. The irrationality of the non-singular cubic, for example, introduces difficulties that can be mastered *simply* and rigorously by the use of Noether's $Af+B\phi$ theorem for cubics and the parametric representation of the curve by means of elliptic functions,—concepts both of which the student will find to be of fundamental importance in more advanced work. For an introduction of this character, the text under consideration is conspicuously well adapted. The presentation of the various topics makes full use of the peculiarities of cubic curves, yet the reasoning involved is of the type that the reader will need in the more difficult study of curves of higher order

The classical discussions of cubic curves in such texts as Salmon, Clebsch-Lindemann, Durege, and Schroeter, were excellent in their time, but have now become somewhat antiquated. In none of them do we find so wise a selection of material nor so balanced an emphasis on the various topics, as in this brief yet comprehensive and readable text.

C. H. SISAM

## REYMOND'S HISTORY OF SCIENCE

*Histoire des Sciences Exactes et Naturelles dans l'Antiquité Gréco-Romaine*, by Arnold Reymond. Paris, Librairie Scientifique Albert Blanchard, 1924. vii+238 pp.

The Preface of this book is from the pen of Léon Brunschvicg, and is followed by an Introduction giving an outline of Babylonian and Egyptian science. The first three chapters are an historical survey of Greek and Roman science. The last six chapters deal with principles and methods in mathematics, astronomy, mechanics, chemistry, natural history, and medicine, as developed in Greece and Rome. In various places, there is pointed out to the reader the connection of ancient conceptions in science with those of later periods.

The work under review is not an independent research based upon the study of original sources, but a compilation from European publicatons. Of assistance to the studious reader are the six pages given to bibliography. Our assurance that all important publications are included in the list is shaken somewhat by the omission of all reference to G. Eneström and his BIBLIOTHECA MATHEMATICA. In the case of the Moscow papyrus, the author overlooked the all-important early Egyptian computation of the volume of the frustrum of a quadrangular pyramid.* Except for a reference to an article of J. H. Breasted that was published in Europe, American scholarship is ignored completely. For a general view point of Babylonian and Egyptian mathematics, a reference to L. C. Karpinski† would have been of value. R. C. Archibald's reconstruction of Euclid's *Divisions of Figures*‡ escaped the attention of the author, as did also D. E. Smith's choice article§ on Greek computation. American publications would have afforded the author a profounder realization of the importance in the history of the theory of limits of Greek discussion of "Indivisibles" and of Zeno's arguments on motion.‖ Of interest would have been the British defence of Aristotle's treatment of falling bodies, to the effect that Aristotle dealt with terminal velocities of a body falling through a resisting medium,¶ and a

* ANCIENT EGYPT, vol. 17, p. 100.

† AMERICAN MATHEMATICAL MONTHLY, vol. 24 (1917), pp. 257–265.

‡ R. C. Archibald, *Euclid's Book on Divisions of Figures*, 1915.

§ D. E. Smith, BIBLIOTHECA MATHEMATICA, (3), vol. 9 (1908–09), pp. 193–195.

‖ F. Cajori, AMERICAN MATHEMATICAL MONTHLY, vol. 22 (1915), pp. 1, 39, 77, 109, 143, 179, 215, 253, 292.

¶ NATURE, vol. 92 (1914), pp. 584, 585, 606.

critique of that defence.* The book under review assists in the perpetuation of the legend, based upon incorrect evidence, that Sir Isaac Newton's delay of twenty years in the announcement of his law of universal gravitation was due to measurements of the size of the earth far below the true value, and that only Picard's determination finally enabled Newton to verify his law. There is very strong evidence in support of the view that Newton's difficulties were of a wholly different character and related to the unsolved problem of the attraction of a spheroid upon an external particle.† In presenting ancient conceptions on the atomic theory and on infinity, C. J. Keyser's article on Lucretius would have afforded illuminating information.‡ Lucretius deserves attention also in the presentation of ancient conceptions of heat.§ But strange to say, Reymond makes only a passing reference to him. In presenting the place of Pliny in the history of science, a reference to the monumental work of Lynn Thorndike‖ would have been very much to the point. It is a bit strange that in discussing the ancient abacus and the lunes of Hippocrates of Chios, Ball's *Short History of Mathematics* should be the only authority cited.

The author's style of exposition is clear. Readers will find the book entertaining and, in general, quite accurate.

FLORIAN CAJORI

## THE ARITHMETIC OF NICOMACHUS

Nicomachus of Gerasa, *Introduction to Arithmetic.* Translated by Martin Luther D'Ooge, with studies in Greek Arithmetic by Frank Egleston Robbins and Louis Charles Karpinski. New York, Macmillan, 1926. x+318 pp. Price $3.50.

This important work appears as volume XVI of the Humanistic Series of the University of Michigan Studies, a series that is doing much to establish and maintain a high standard of scholarship in this country. The translation was made by the late Professor D'Ooge, whose death eleven years ago was the occasion of a great loss not merely to his university but to the cause of classical scholarship everywhere. It had been completed at the time of his death, but the "supporting studies," as the editors call them, were undertaken later by Professors Robbins and Karpinski. Professor Robbins contributed chapters on the development of Greek arithmetic before Nicomachus; on the latter's life, works, and philosophy; on his philosophy of number, his translators and commentators; on the manuscripts and texts of his works; and on his language and style. Professor Karpinski's contributions consist of chapters on the sources of Greek

* SCHOOL SCIENCE AND MATHEMATICS, vol. 21 (1921), p. 638 ff.

† W. W. R. Ball, *An Essay on Newton's Principia*, 1893, p. 7; see also the ARCHIVIO DI STORIA DELLA SCIENZA, vol. 3 (1922), pp. 201–204.

‡ C. J. Keyser, this BULLETIN, vol. 24 (1918), p. 321.

§ ISIS, vol. 4 (1922), p. 483–492.

‖ *History of Magic and Experimental Science*, vol. 1 (1923), pp. 42–99.

mathematics, on the contents of the *mathematica*, on the Greek numerals, on the successors of Nicomachus (part of the chapter), and on a certain theorem found in the *Introductio*.

It is no disparagement of the scholarship displayed in the "supporting studies" to say that the importance of the work will doubtless be judged to lie largely in the translation itself, the first complete one in our language, and one long needed. The treatise is so important in its bearing upon the history of mathematics and, indeed, upon the history of human thought, that it should long since have been made available to non-classical students. On the other hand, it is a matter of congratulation that the translation was not undertaken by one less worthy of the task, and men are rarely found who are as gifted as Professor D'Ooge in the knowledge of Greek and in the ability to express the delicate shades of that language in equally delicate English.

Nicomachus did for the ancient theory of numbers what Euclid had already done for elementary geometry and Apollonius for conics,—he systematized the knowledge of his predecessors and expressed the result with simplicity of language and with a clearness that appealed to his people. Through Boethius, who embodied in a popular Latin treatise part of the theory thus set forth in Greek, his influence dominated the teaching of the subject for more than fourteen centuries.

The translation itself is, like Jowett's *Plato*, a delightful piece of English, revealing not a little of the power and literary style of the Greek. Only one who was imbued with the spirit of the language could have expressed himself so simply and so directly. The editors do not tell us whether the notes are due to Professor D'Ooge or to themselves, but in either case they are models of what should be expected in a work of this kind, not so much in elucidating the text as in giving valuable information relating to technical terms and to the works of such writers as Euclid, Theon of Smyrna, and Iamblichus.

In the studies the reader will find evidence of a large amount of scholarly research,—to use a term "defamed by every charlatan and soiled with all ignoble use,"—and he will be particularly indebted to the authors for valuable information relating to the life and works of Nicomachus. Naturally, however, he will find a considerable diversity of style, due to the fact that the treatise represents no less than three collaborators whose diction and taste vary to a noticeable degree. He will be quite sure that Professor D'Ooge would not have used both "demonstrative geometry" and "demonstrational geometry" (p. 4) in the same chapter, even with the half-hearted support of Murray, nor the obsolescent spelling "parallelopiped" (p. 57), nor both the forms "Hero" and "Heron," nor both "arithmetic" and "arithmetical" as adjectives. The use of the form *Abbaci* in speaking of the *Liber Abaci* of Fibonacci, simply because the Italian title page gave a form which Leonardo did not use in his *Incipit*, will also strike the reader as foreign to Professor D'Ooge's style and method.

On the other side, readers who are conversant with the history of mathematics will regret a tendency which is frequently and painfully apparent to make sweeping assertions that are not confirmed by the con-

text or by the notes, and which are open to serious doubt. A few of these will serve to illustrate their nature.

P. 4. In the "ἀριθμητική of which the arithmetic of Nicomachus is a specimen, . . . . we have . . . . the forms of proof and the rigor of the demonstrational (*sic*) geometry." Certainly the forms of proof are entirely different from those of Euclid, and few would place them in the same class for logical rigor.

P. 5. "For the sources of the early Greek arithmetical sciences we must look to Egypt and Babylon, possibly even beyond to India and China." If this means anything definite, it means that traces of the Greek number theory as developed by the Pythagoreans are to be found there, but certainly there is nothing in the text or the notes to justify any such assertion. The mere fact that people could count, could write numbers, and could recognize odd and even numbers at a very early period, as well as use simple problems in series, affords not the slightest historical evidence of any concept of the Greek number theory, as the term is commonly understood.

P. 9. In speaking of "early Egyptian mathematical science" the assertion is made that the "few surviving documents give indications of development along many different lines of mathematical thought." A reader might legitimately look for some information as to the documents referred to, and as to the "many different" topics. There may occur to him a few rhetorical equations, the weak mensuration work of Ahmes, and the probable truncated pyramid in the Moscow papyrus, but what are the "*many different lines*" discussed in the "documents"? The doubt cannot be removed by the other sweeping statement about "further *definite* indications of *real progress* in mathematical *thinking* among the Egyptians." The reader is certainly justified in asking for some evidence that will stand scrutiny, even in the form of note references.

Pp. 10, 11. The statement is made that "the most notable advance in astronomy in Babylon was undoubtedly made during the period in which the science was making real progress in Greece." Since this "real progress" was presumably made in the period from Aristarchus (*c.* 270 B.C.) to Ptolemy (*c.* 150 A.D.), it will be interesting to know the details of the "most notable advance in Babylon," even if this term is taken to mean Babylonia, and Babylonia is taken to mean Chaldea, as is quite proper. Just what "most notable advance" was made after the Persian conquest of 539 B.C., which was the period of Pythagoras; or after the conquest by Alexander, which also antedated Aristarchus? If on the other hand, the "*real* progress" in Greek astronomy was made before Aristarchus, who made it, and what was the date of "the most notable advance in astronomy in Babylon"?

The above are merely typical of questions that arise in connection with various general assertions in the text. A few further notes are given below as typical of certain minor questions that will occur to nearly every reader.

P. 66. The use of the term "geometrical numerals," referring to the early Minoan type.

The validity of the *ex cathedra* statement: "Heath's assertion that we reckon 'with words' is not correct." It might interest a psychologist who was explaining various types of images.

The implied contradiction of Heath's opinion that the Greek notation "did not adversely affect their arithmetic." It would be interesting to see the argument that it really did affect their *arithmetica*.

P. 67. The statement that the short-legged form of Π came after the isosceles form.

P. 68. The use of small modern Greek letters and accents instead of forms more nearly like those found in the early inscriptions and manuscripts.

The use of σαμπῖ as if it were the ancient Greek name for the character for 900.

Pp. 138–140. The indiscriminate use of Isidore and Isidorus, without any apparent reason.

P. 145. The use of San Sepulchri for San Sepolchro or San Sepolcro, or for Santi Sepulchri in case the Latin genitive is given with *Prefatio* as in the 1494 edition. The form San Sepulchri is neither Italian nor Latin.

The spelling Marg*h*arita Philosophica.

It may be allowable to suggest also, that the force of certain passages would have been greater without such expressions as "I hold that" (p. 12), "it seemed to me" (p. 289), and "This gives my theorem" (p. 290). As to this latter interesting relation, it would not have been claimed by the writer of the above statement about "my theorem" if he had looked more fully into the history of the subject. Mr. Jekuthial Ginsburg, who has done considerable work in the history of number theory, has called the reviewer's attention to the fact that Bretschneider not only gave it over eighty years ago (Grunert's ARCHIV, I, 415) but extended it to even powers as well, saying: "Turner's Theorem is a special case of a far-reaching general theorem which I discovered years ago but did not consider new." The name "Turner's Theorem," suggested by Sir William Rowan Hamilton, was ridiculed by Boncompagni, Turner having merely proved the statement made by Nicomachus, as others had done before him. Bretschneider showed, as part of his general theorem, that every odd power $2n+1$ of an integer $p$ is equal to the sum of $p^n$ consecutive odd numbers beginning with $p^n(p-1)+1$. In particular, in the case of seventh powers, as given in the article under review, $n=3$. Then, for example, for $4^7$ we have $p=4$, and so the sum of $4^3$, or 64, consecutive odd numbers beginning with $4^3(4-1)+1$, or 193, is $4^7$. In other words, this well known theorem is precisely the one now claimed as a new discovery,—as, indeed, was very likely the case.

DAVID EUGENE SMITH

# SHORTER NOTICES

*Grundbegriffe und Grundprobleme der Korrelationstheorie.* By A. A. Tschuprow. Leipzig-Berlin, Teubner, 1925. v+153 pp.

This book presents in extended form a series of lectures given in the insurance seminar of the University of Christiania. The main purpose of the book is to give a unified treatment of correlation theory with special reference to the fundamental conceptions and logical foundations of the theory. It seems to be very properly held that the treatment of the logical foundations of the method of correlation has not kept pace with the wide range of applications. The exposition does not proceed from the standpoint of the analysis of numerical data, but from the standpoint of a priori probability. The theory of correlation is regarded as an organic part of the theory of probability. The treatment seems fairly well described as an idealization of the somewhat empirical concepts of the English school of statisticians by a sharper formulation of definitions and underlying concepts.

Much is made of an expressive phraseology involving the concepts of chance variable and stochastic connection. A chance variable of order $k$ is defined as a variable which takes any one of $k$ values with assigned probabilities. For example, the number that will be thrown with a die in a single throw is a chance variable. When $x$ is assigned, and $y$ is a corresponding chance variable which takes values with definite probabilities, there is said to be a stochastic connection between $x$ and $y$. For example, if in throwing two dice, the first gives a value $x=3$, then the corresponding total $y$ for the two dice is $y=4, 5, 6, 7, 8,$ or 9, and there is a stochastic connection between the chance variables $x$ and $y$. Much is made of the conception of stochastic dependence as distinguished from the more familiar conception of the functional dependence of two variables. In fact, the recognition of a clear distinction between the conceptions of stochastic connection and functional dependence constitutes a first step in following the exposition in this book.

The explanation of the stochastic connection of $y$ with $x$ follows the regression method, and calls for a complete characterization of the theoretical array (das bedingte Verteilungsgesetz) of $y$'s for any assigned $x$. While this conception is quite an advance over the early Pearson concept that $y$ is correlated with $x$ when the mean values of the theoretical arrays of $y$'s are not constant but are functions of $x$, the author has hardly given adequate credit to Pearson for a much more general view of correlation given in DRAPERS' COMPANY RESEARCH MEMOIRS, Biometric Series II (1905), p. 9. In this more general view we may say that whenever any characteristic of the theoretical arrays of $y$'s changes from one assigned value of $x$ to another, there is a stochastic dependence.

For the characterization of arrays, it is assumed that moments and product moments will serve to determine the necessary parameters for a complete characterization of arrays.

One of the most important parts of the book is concerned with estimating the a priori values of correlation coefficients, correlation ratios, and other statistical constants from the corresponding empirical values. In this part of the work, there is given a careful treatment of the sampling problems involved.

Taken as a whole, the reviewer considers that the book is an important contribution to more critical and rigorous thinking on the methods and theory of correlation.

H. L. RIETZ

*The Theory of Measurements.* By Lucius Tuttle and John Satterly. London, Longmans, Green and Co., 1925. xii+333 pp. $4.50.

For readers who are acquainted with the book which has the same title as the one now under review and which was published by Tuttle in the year 1916, the following comments will doubtless be adequate. A page by page comparison showed that Chapters I to XX of the later volume are very nearly the same as the entire text of the original book. The changes consist chiefly in the addition of a few diagrams and articles, and in the correction of errors of statement. A new chapter, XXI, deals with applications to biology. It involves (*A*) frequency problems and (*B*) illustrations of the use of graphic analysis and mathematical equations in biological investigations. The last chapter, XXII, is devoted to the practical evaluation of plane areas. It includes interesting paragraphs on the hatchet planimeter. The value of the entire book is enhanced by the addition of nearly 500 examples for solution by the student. The appendix of useful numerical tables is preceded by answers to all the quantitative examples. Satterly's collaboration has improved the text very appreciably and the only adverse criticism worthy of note is that the number of typographical errors is sufficiently large to annoy a particular student.

To readers who have not seen the unrevised text the following supplementary statements may be addressed. The object of the book is to train students to criticise the instruments by which their readings are taken, the accuracy of their observations, and the methods on which their observations are planned and by which they are reduced. The first twenty chapters are devoted to weights and measures, angles and circular functions, significant figures, logarithms, small magnitudes, slide rule, graphic representation, graphic analysis, interpolation and extrapolation, coordinates in three dimensions, accuracy, the principle of coincidence, errors and statistical methods, deviation and dispersion, weighting of observations, rejection criteria, indirect measurements, least squares, systematic and constant errors. The book may be used in connection with courses of mathematics as well as for courses in physics, and for this reason the requirements of the mathematician have been especially kept in mind during the preparation of the text. No knowledge of trigonometry is presupposed

and the course is progressively graded in difficulty. The book in its present form is reliable and up to date.

H. S. UHLER

*Proceedings of the First International Congress for Applied Mechanics*, edited by B. Biezeno and J. M. Burgers. Delft, 1924. xxii+460 pp.

In September, 1922, a conference of scientists from various countries assembled at Innsbruck to discuss questions of hydrodynamics and aerodynamics. At that meeting the idea originated of trying to call together a larger conference to consider questions ranging over the whole domain of applied mechanics. This congress met during April 1924 in Holland at Delft, the seat of the Dutch Technical University. During the conference it was decided to establish a permanent institution called the International Congress for Applied Mechanics, to assemble every fourth year. To get a difference of phase of two years with the International Congress of Mathematics the next meeting (at Zürich) will be held in the autumn of 1926. General matters were left in charge of an International Congress Committee on which the United States is represented by Professor J. S. Ames of Johns Hopkins.

In arranging for the congress at Delft two days were assigned for general meetings devoted to review lectures on subjects in which considerable advance had been made during the last few years. About a dozen topics were selected and a number of prominent scientists invited to treat them. These lectures, somewhat abridged by the editors, occupy the first 176 pages of the *Proceedings*. The remainder of the volume contains original communications divided into three sections, rational mechanics, theory of elasticity, hydro- and aerodynamics. Any attempt to summarize the contents of the nearly 40 papers in this group could hardly be more than a mention of titles. While a few are concerned with graphical methods or reports of experimental work, most of them contain investigations of a mathematical nature.

H. B. PHILLIPS

*Introduction à la Gravifique einsteinienne*. By Th. de. Donder. MÉMORIAL DES SCIENCES MATHÉMATIQUES, No. 8. Paris, Gauthier-Villas, 1925. 56 pp.

This little pamphlet is the first of a series of three by the same author to be published in the same series of semi-popular expositions. The other two will be entitled *Théorie des Champs Gravifiques* and *Quelques Problèmes de la Gravifique*. In the introductory pamphlet now before us the author first presents an excellent development of the generalized theory of relativity. De Donder then applies the general theory to the task of deriving the restricted theory in a novel and most interesting manner. The effect of this treatment upon my own consciousness was quite as stimulating as was, several years ago, the presentation of the definition of an integral before that of a derivative in Osgood's *Funktionentheorie*. I believe that others may find this book equally stimulating.

C. N. REYNOLDS, JR.

*Statistical Analysis*. By Edmund E. Day. New York, Macmillan, 1925. xxvii+459 pp.

*Statistical Analysis* is one of a number of recent excellent treatises on statistical methods written by professors of economics or statistics. These books, which include Mill's *Statistical Methods*, Jerome's *Statistical Method*, and Chaddock's *Principle and Methods of Statistics* and perhaps others that have not come to my notice, mark an advance over the older books. The new books are in general larger and great pains have been taken to make the pages pleasing to the eye, as well as to present simply and clearly the fundamentals of the subject. While not professing to enter into mathematical intricacies, they nevertheless increase somewhat the mathematical content offered in a first course in statistics to students of economics. And they give in extremely attractive form that groundwork of statistics which is above all essential. They do not contain as much mathematics as Gavett's *First Course in Statistical Method*, Kelley's *Statistical Method*, or Forsyth's *Mathematical Analysis of Statistics*.

Professor Day is Professor of Economics and Dean of the School of Business Administration at the University of Michigan. In *Statistical Analysis* his chief interest lies in the analysis and interpretation of statistical data. The collection and tabulation of data is treated in appendices. In an appendix, also, is the report of the Joint Committee on Standards for Graphic Presentation. Graphical representation is, indeed, regarded as a useful adjunct, and is used freely throughout the text; but it is not treated as an end in itself. The technique and manipulation of figures is also regarded as secondary. Nevertheless, complete forms are exhibited—including, for example, the calculation of the correlation coefficient, singly and jointly with the correlation ratio; marked—"Source: A. R. Crathorne, '*Calculation of the Correlation Ratio*,' JOURNAL OF AMERICAN STATISTICAL ASSOCIATION, Sept., 1922, pp. 394–396." Professor Day regards statistics as primarily a form of inductive logic in which all data bearing upon a problem must be given careful consideration, and in which quantitative results furnish valuable but not infallible or conclusive evidence.

The topics include classification, frequency, spatial, and temporal distributions, averages, variability, skewness, correlation coefficient and ratio, regression lines, correlation from ranks, time series with increments and rates of change, secular trend, seasonal variation, cyclical and irregular fluctuations, lag, and index numbers. In the appendix are tables of logarithms, squares, square roots, and reciprocals, and the relationship between $r$ and $\rho$, a list of general references, and an index.

E. L. DODD

# NOTES

It has been decided again to increase the size of this BULLETIN, so that the present volume will contain approximately 750 pages. The size of the present number results from this decision.

At the Annual Meeting of the Society, to be held in Philadelphia, December 28–30, 1926, in affiliation with the American Association for the Advancement of Science, Professor G. D. Birkhoff will deliver his retiring presidential address; it will be entitled *A mathematical critique of some physical theories*. The Josiah Willard Gibbs Lecture will be delivered by Professor H. B. Williams, of the College of Physicians and Surgeons of Columbia University, on the subject *Mathematics and the biological sciences*. On Thursday afternoon, Dec. 30, there will be held a joint meeting of this society, The Mathematical Association of America, and Section A of the American Association for the Advancement of Science, at which Professor W. H. Roever will deliver his retiring adderss as Vice-President of Section A on the subject *The weight field of force of the earth*, and Professor F. D. Murnaghan, of the Johns Hopkins University, will speak on the topic *The duty of exposition, with special reference to the Cauchy-Heaviside theorem*. An informal dinner for the mathematicians and their guests will be held at the Aldine Hotel on Wednesday evening.

The reports and addresses that have appeared or are appearing in the present volume of this BULLETIN are now available as separate reprints, and may be ordered at the office of the Society in New York. Those now on hand are: *Properties of unrestricted functions*, by Henry Blumberg, pages 132–148; *Some modern views of space* (The Gibbs Lecture for 1925), by James Pierpont, pages 225–258; *The Borel theorem and its generalizations*, by T. H. Hildebrandt, pages 423–474; *Some recent work in the calculus of variations*, by Arnold Dresden, pages 475–521. These will be sold practically at cost at prices announced on a fly-leaf of this issue; in general the charge will be less than one cent per page.

Professors E. W. Chittenden and E. B. Stouffer have been appointed associate editors of the TRANSACTIONS OF THIS SOCIETY.

The second number of volume 28 of the TRANSACTIONS OF THIS SOCIETY contains the following papers: *New division algebras*, by L. E. Dickson; *The first and second variations of a double integral for the case of variable limits*, by H. A. Simmons; *On extending a continuous* (1–1) *correspondence of two plane continuous curves to a correspondence of their planes*, by H. M. Gehman; *Systems of equations in an infinity of unknowns, whose solution involves an arbitrary parameter*, by I. M. Sheffer; *Solution of certain functional equations relative to a general linear set*, by M. H. Ingraham; *Combinatorial analysis situs*, by J. W. Alexander; *Geometries of paths for which the equations of the paths admit* $n(n+1)/2$ *independent linear first integrals*, by L. P. Eisenhart; *Multiply transitive substitution groups*, by G. A. Miller; *An extension of Lagrange's expansion*, by H. Bateman.

The second number of volume 48 of the AMERICAN JOURNAL OF MATHEMATICS (April, 1926) contains: *The theory of the binary octavic*, by A. M. Whelan; *The Borel summability of Fourier series*, by M. H. Stone; *Self dual space curves*, by T. R. Hollcroft; *Expansions in terms of certain polynomials connected with the gamma-function*, by B. P. Hoover; *On the value of the Napierian base*, by D. H. Lehmer; *On differential inversive geometry*, by F. Morley.

The third number of volume 27, series 2, of the ANNALS OF MATHEMATICS (March, 1926) contains: *Transformation of the Kummer criteria in connection with Fermat's last theorem*, by H. S. Vandiver; *Algebraic potential curves*, by J. L. Coolidge; *A method of deriving the infinite double products in the theory of elliptic functions from the multiplication theorems*, by G. Mittag-Leffler; *Miscellaneous questions in the theory of differential equations. I. On the method of Frobenius*, by E. Hille; *Non-measurable functions connected with certain functional equations*, by H. Blumberg; *Entire functions defined by certain power series*, by J. T. Colpitts; *Approximation of curves and surfaces by algebraic curves and surfaces*, by P. A. Smith; *Note on matrices in a given field*, by J. H. M. Wedderburn; *On the arithmetical applications of the power series for theta quotients*, by E. T. Bell; *Conditions for self dual curves*, by T. R. Hollcroft; *Semi-parallel maps of surfaces*, by W. C. Graustein; *Projective invariants of affine geometry of paths*, by O. Veblen and J. M. Thomas.

The Faculty of Science of the Imperial University of Tokyo has begun publication of a separate section (Section I) of its JOURNAL, to be devoted to mathematics, astronomy, physics, and chemistry.

The second International Congress for Applied Mechanics was held at Zurich, September 12–18, 1926. The first of this series of congresses was held at Delft in 1924.

An anniversary volume entitled *The Work of the College Entrance Examination Board*, which has just been published by that Board through Ginn and Company, contains a history of mathematical requirements for college entrance since the time (1892) of the famous Committee of Ten. Among members of this Society who have rendered conspicuous service are Professor T. S. Fiske, who has been Secretary of the Board since 1901 when he succeeded President Butler, Professor W. F. Osgood, Professor Dunham Jackson, and members of several committees of this Society, of the Mathematical Association, and of the Board itself. In the twenty-five years covered by this volume, the number of candidates examined in one year has increased from less than one thousand to over twenty thousand.

Professor Maurice Fréchet, of the University of Strasbourg, is preparing Part 3 (*Modern Theoretical Researches*) of Volume 1 of Borel's *Traité du Calcul des Probabilités*. In order to avoid oversights, he requests that notice be sent to him regarding results on probabilities, other than applications, which deserve mention and which have not been noted in Parts 1 and 2 of Volume 1. Precise references, and even brief summaries, would be welcomed, so that the work may be as complete as possible.

Professor W. C. Graustein has presented to the Sociéte Mathématique de Belgique the amount of the prize that he received from the Royal Academy of Belgium for his memoir entitled *Méthodes invariantes dans la géométrie infinitésimale des surfaces* (see this BULLETIN, vol. 32, p. 177), with the request that it be used to found a prize. This prize, which will be known as the Prix Graustein, will be awarded for an important contribution to infinitesimal geometry; competition is restricted to the United States and Belgium. Competing memoirs should be sent to the Société Mathématique de Belgique, rue d'Egmont 11, Brussels, before October 15, 1927.

The Academy of Sciences of Naples announces the following subject for a prize memoir: To expound and coordinate recent researches on the concept of parallelism in a curved space, and to contribute to the theory. Competing memoirs should reach the Academy by May 31, 1928.

The Royal Society of Edinburgh has awarded its Keith Prize for the period 1923–25 to Professor H. W. Turnbull, of St. Andrews, for his papers on hyper-algebra, invariant theory, and algebraic geometry, published in the PROCEEDINGS OF THE ROYAL SOCIETY OF EDINBURGH.

Cambridge University has awarded its Mayhew Prize in applied mathematics jointly to J. A. Gaunt, of Trinity College, and A. H. Wilson, of Emanuel College.

The medal of the American Society of Mechanical Engineers has been awarded to Professor R. A. Millikan, of the California Institute of Technology.

Professor N. E. Nörlund, of the University of Copenhagen, has been elected corresponding member of the Paris Academy of Sciences in the section of geometry, as successor to the late C. Guichard.

The Bavarian Academy of Sciences has elected as corresponding members Professor Harald Bohr, of the Copenhagen Technical Institute, and Professor Niels Bohr, of the University of Copenhagen.

Professors G. H. Hardy and C. V. L. Charlier have been elected corresponding members of the Vienna Academy of Sciences.

Sir Ernest Rutherford, Professor H. A. Lorentz, and Dr. H. LeChatelier have been elected foreign members of the Académie Polonaise des Sciences.

Professor Paul Langevin, of the Collège de France, has been elected president of the French Association for the Advancement of Science.

Professor Salvatore Pincherle has been elected president of the Unione Matematica Italiana.

Professors Niels Bohr, Max Planck, and Arnold Sommerfeld have been elected foreign members of the Royal Society of London.

Cambridge University conferred the degree of doctor of science on Sir Ernest Rutherford, on the occasion of the celebration by Trinity College of the Bacon Tercentenary.

The University of St. Andrews has conferred an honorary doctorate on Professor E. T. Whittaker, of the University of Edinburgh.

Columbia University, Yale University, and the University of Wisconsin have conferred honorary doctorates on President Max Mason, of the University of Chicago.

Columbia University has conferred an honorary degree on Dr. W. S. Adams, director of Mount Wilson Observatory.

Dr. William Bowie, of the United States Coast and Geodetic Survey, has been elected a member of the Norwegian Academy of Sciences.

Professor R. A. Millikan has been elected a foreign member of the Göttingen Academy of Sciences.

Professor M. I. Pupin, of Columbia University, has received an honorary degree from Rutgers University.

The University of Wisconsin has conferred an honorary degree on Professor A. N. Whitehead, of Harvard University.

National Research Fellowships in mathematics have been awarded as follows: at the January, 1926, meeting to C. E. Hille, B. O. Koopman, G. M. Merriman, D. E. Richmond, D. V. Widder; at the April, 1926, meeting to R. W. Barnard, L. R. Ford, Orrin Frink, R. G. Lubben, C. F. Roos, P. A. Smith, Louis Weisner.

Dr. Louis Antoine has been promoted to a professorship of calculus at the University of Rennes.

Dr. René Milloux has been appointed *maître de conférences* in mathematics at the University of Strasbourg.

Professor M. Riesz, of Stockholm, has been appointed to a professorship at the University of Lund.

Dr. Rudolph Weyrich, of the University of Marburg, has been appointed professor of mathematics at the German Technical School at Brunn.

Dr. Paul Zühlke has been appointed honorary professor of the teaching of mathematics at the University of Marburg.

The late W. W. Rouse Ball, of Trinity College, has bequeathed the sum of £25,000 to Cambridge University to endow a professorship or readership in mathematics, expressing the hope that incumbents of this professorship or readership should include in their lectures historical and philosophical aspects of the subject.

Dr. W. E. H. Berwick, of the University of Leeds, has been appointed professor of mathematics at the University of North Wales, Bangor.

Dr. E. L. Ince, of the University of Liverpool, has been appointed professor of pure mathematics at the Egyptian University of Cairo.

Dr. J. P. Ballantine, of Columbia University, has been appointed assistant professor of mathematics at the University of Washington.

Mr. J. E. Burnam has been appointed professor of mathematics at Simmons College, Abilene, Texas.

Assistant Professor G. M. Conwell, of the New York State College for Teachers, has been promoted to a full professorship of mathematics.

Dr. Tobias Dantzig has been appointed assistant professor of mathematics and mechanics at the University of Maryland.

Professor J. E. Davis, of Pennsylvania State College, has been appointed associate professor of mathematics at Drexel Institute.

Professor L. S. Dederick, of the University of British Columbia, has been appointed professor of mathematics at St. Stephen's College.

Professor W. C. Eells, of Whitman College, has been granted leave of absence for the academic year 1926–27 to serve as research assistant in educational measurements at Stanford University.

Mr. Ira Gwinn has been appointed to an assistant professorship at Morningside College.

Dr. R. W. Hartley has been appointed professor of mathematics at Southwestern College, Memphis.

Assistant Professor Gertrude A. Herr, of Iowa State Teachers College, has been promoted to an associate professorship of mathematics.

Dr. G. W. Hess has been appointed associate professor of mathematics at Howard College, Birmingham.

Professor R. A. Johnson, of Hamline University, has been appointed assistant professor of mathematics at Hunter College.

Associate Professor A. F. Kovarik has been promoted to a full professorship of physics at Yale University.

Dr. K. W. Lamson, of Columbia University, has been appointed assistant professor of mathematics at Lehigh University.

Dr. B. Z. Linfield has been appointed associate professor of mathematics at the University of Virginia.

Dr. W. L. Machmer, of the department of mathematics at the Massachusetts Agricultural College, has been made dean of that College.

Professor F. W. Owens, of Cornell University, has been appointed head of the department of mathematics at Pennsylvania State College.

Dr. B. C. Patterson has been appointed assistant professor of mathematics at Washington and Jefferson College.

Professor Georgia E. Robinson, of Shorter College, has been appointed professor of mathematics at Jefferson City Junior College.

Mr. H. A. Robinson has been appointed head of the department of mathematics at Agnes Scott College.

Miss Jessie M. Short has been promoted to an assistant professorship of mathematics at Reed College.

Dr. W. G. Simon has been promoted to a full professorship of mathematics at Adelbert College, Western Reserve University.

Associate Professor L. L. Smail, of the University of Texas, has been appointed associate professor of mathematics at Lehigh University.

Dr. W. H. Taylor has been appointed head of the department of mathematics at Hamline University.

Professor H. S. Uhler, of Gettysburg College, has been appointed associate professor of physics at Sheffield Scientific School, Yale University.

Assistant Professor J. K. Whittemore has been promoted at an associate professorship of mathematics at Yale University.

The following appointments to instructorships are announced: Cornell University, Mr. W. T. MacCreadie; New York University (Washington Square College), Dr. Fay Farnum; Rutgers University, Mr. M. G. Galbraith; Vassar College, Miss Julia W. Bower; State College of Washington, Mr. S. A. Schelkunoff; Yale University, Dr. H. M. Gehman.

Professor Giovanni Frattini, of the Technical Institute and the Military College at Rome, died in 1925 at the age of seventy-three.

Dr. J. T. Bottomley, of the University of Glasgow, died May 18, 1926. at the age of eighty-one.

Professor W. J. Lewis, of Cambridge University, known for his work in geometrical crystallography, died April 16, 1926, at the age of seventy-nine.

Professor V. A. Stekloff, of the University of Leningrad, died May 30, 1926, at the age of sixty-three.

Professor B. E. Carter, of Colby College, died June 11, 1926, at the age of sixty. Professor Carter had been a member of the American Mathematical Society since 1906.

Colonel H. H. Ludlow, U. S. A., died August 14, 1926. Colonel Ludlow had been a member of the American Mathematical Society since 1896, and was a life member.

Professor Gertrude W. Mendenhall, of the North Carolina College for Women, died in April, 1926. Professor Mendenhall was a member of the American Mathematical Society.

# NEW PUBLICATIONS

## PART 1. PURE MATHEMATICS

BAGCHI (H.). A course of geometrical analysis. Calcutta, Chuckervertty, Chatterji and Company, 1926. 8+562 pp.

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BLASCHKE (W.). See KLEIN (F.).

BOON (F. C.). Puzzle papers in arithmetic. London, Mills and Boon, 1925. 55 pp.

BRAUER (R.). Ueber die Darstellung der Drehungsgruppe durch Gruppen linearer Substitutionen. (Dissertation.) Göttingen, Dieterichsche Universitäts-Buchdruckerei, 1925. 72 pp.

BUHL (A.). Formules Stokiennes. (Mémorial des Sciences Mathématiques, No. 16.) Paris, Gauthier-Villars, 1926. 60 pp.

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ČECH (E.). See FUBINI (G.).

CHICAGO. University of Chicago, Abstracts of theses. Volume 1: 1922–1923. Chicago, University of Chicago Press, 1925. 500 pp.

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CUNNINGHAM (A. J. C.). Binomial factorisations. Volume 7. London, Hodgson, 1925. 117 pp.

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D'OOGE (M. L.). See NICOMACHUS.

ENRIQUES (E.). L'évolution de la logique. Traduit par G. E. Monod-Herzen. Paris, Chiron, 1926. 204 pp.

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HAYASHI (K.). Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen und deren Produkte sowie der Gammafunktion. Berlin, Springer, 1926. 6+284 pp.

HEIJTING (A.). Intuitionistische axiomatiek der projektieve meetkunde. (Dissertation, Amsterdam.) Groningen, Noordhoff, 1925. 92 pp.

HELFENSTEIN (A.). Das Wesen der Stoffwelt. Leipzig, Deuticke, 1925. 2+94 pp.

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